Complex Burgers Equation: A probabilistic perspective

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Abstract

In 1997 Yves LeJan and Alain-Sol Sznitman provided a probabilistic gateway in the form of a stochastic cascade model for the treatment of 3d incompressible, Navier-Stokes equations in free space. The equations themselves are noteworthy for the inherent mathematical challenges that they pose to proving existence, uniqueness and regularity of solutions. The main goal of the present article is to illustrate and explore the LeJan-Sznitman cascade in the context of a simpler quasi-linear pde, namely the complex Burgers equation. In addition to providing some unexpected results about these equations, consideration of mean-field models suggests analysis of branching random walks having naturally imposed time delays.

1 Introduction

Throughout his career Chuck Newman has demonstrated mathematical insights that are remarkable for both their depth and their wide ranging relevance to seemingly diverse areas of mathematics and science. There are many things to admire about the way in which Chuck is able to resolve things mathematically. The following story may be a lesser known example, but it is relevant to the topic of this article, and is mainly being shared in admiring tribute to Chuck. During the early 1980's while still a professor at the University of Arizona, Chuck attended a colloquium talk on fluids featuring a discussion of Burgers equation. As some of us were leaving the talk, Chuck shared a scrap of paper in which he had doodled[1] a striking connection between Burgers equation for fluids and one of his first loves, statistical physics. Specifically, in a matter of minutes, he had discovered that the magnetization in a finite volume Curie-Weiss probability model is governed, as a function of inverse temperature and external field, exactly by the same Burgers equation presented by the

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[1]The reference to "doodle"is deliberate. Once, when asked how he discovered his central limit theorem for associated random variables given in [40], Chuck replied that he had expected for some time that a central limit theorem should be possible for ferromagnetic Ising models at high temperatures as a consequence of correlation decay: "Then, one day I was doodling with the FKG inequalities and out popped just the right correlation inequalities for the characteristic function of the magnetization."This would prove that central limit theorem.

speaker in a seemingly unrelated context of fluid flow. In addition to providing a rather unique perspective on shocks in quasilinear pde's in relation to spontaneous magnetization, Chuck had quite cleverly provided new insights into a basic model of interest to mathematical physics and probability; see [41]. In particular, early representations of solutions to the Burgers' equation as expected values in an interesting probability model can just as well be attributed to Chuck Newman.

Proposition 1.1 (Newman,1986)**.** *Let* $m = m_n(h,\beta) = \mathbb{E}S_1$ *where* $\{S_i : i = 1,\dots,n\}$ *has the joint distribution* $Z^{-1}\exp[\frac{\beta}{2}\sum J_{ij}s_is_j + \sum h_is_i]\prod\rho(ds_i)$ *with* $J_{ij} = J/2n, (J > 0), h_i = h$ *for all* i, j *and* $\int\exp(Ks^2)\rho(ds) < \infty$ *for all* $K > 0$. *Then*

$$\partial m/\partial\beta = Jm\frac{\partial m}{\partial h} + \frac{J}{2n}\frac{\partial^2 m}{\partial h^2}, \quad m(h,0) = \frac{d}{dh}\ln\int e^{hs}\rho(ds).$$

As remarked in [41], in the usual spin-1/2 Ising models the measure ρ is given by $\rho(ds) = [\delta(s-1)+\delta(s+1)]\frac{ds}{2}$, and the resulting initial condition is $m(h,0) = \tanh(h)$. The classic space-time Burgers equation results by defining $t = \beta J$, $x = -h$ and $\nu = 1/n$. Considerations of complex h arise naturally in ([41], Theorem 6), see also [39, 42], in connection with Chuck's take on the zeros of the partition function and the Riemann hypothesis. As will be seen, the complexification of Burgers suggested by the LJS-cascade is in the initial data and solutions, rather than their spatial domain.

The (unforced) three-dimensional incompressible Navier-Stokes equations governing fluid velocity $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and (scalar) pressure $p(x,t)$, in free space $x\in\mathbb{R}^3, t\geq 0$, are given for initial data v_0 and viscosity parameter $\nu > 0$, by a system of quasi-linear partial differential equations

$$\frac{\partial v}{\partial t} + v\cdot\nabla v = \nu\Delta v - \nabla p, \quad \nabla\cdot v = 0, \quad v(x,0^+) = v_0(x),\ x\in\mathbb{R}^3,\ t>0. \tag{1.1}$$

The equation $\nabla\cdot v = 0$ defines the incompressibility condition. The nonlinear term $v\cdot\nabla v$ is the result of representing the flow in a Lagrangian coordinate system; i.e., $\frac{\partial v}{\partial t}+v\cdot\nabla v$ is the acceleration in a frame following a moving particle and, as such, mathematically intrinsic to the equations.

Like the Riemann hypothesis, settling the question about the global existence of smooth solutions for smooth initial data ranks among the millennial problems for mathematics. The following is the precise Clay prize formulation of a positive resolution provided by Charles Fefferman [19]

Navier-Stokes Milennial Problem (MP): For divergence-free $v_0\in\mathcal{S}$ of rapid decay, show that there exists $p, v\in C^\infty(\mathbb{R}^3\times[0,\infty))$ satisfying (1.1), and the finite energy condition that, for some $C>0$,

$$\int_{\mathbb{R}^3}|v(x,t)|^2dx \leq C, \quad \forall t\geq 0.$$

It may be noted that uniqueness is implicit in this formulation of the problem. Also an alternative formulation posed on the torus with periodic boundary conditions can be made in place of the free space formulation. Of course a negative formulation is also possible. Since Fourier transform is a homeomorphism on the Schwarz space $\mathcal{S}$, an equivalent condition follows accordingly for the initial data.

The goal of the present article is to illustrate and explore a probabilistic gateway to three-dimensional incompressible Navier-Stokes equations (1.1) in Fourier space discovered by Yves

Le Jan and Alain-Sol Sznitman [36], hereafter referred to as the LJS-cascade, but in the context of stochastic cascades derived from the ostensibly simpler complex Burgers equation, and related mean-field cascades to be defined. This will also give an opportunity to provide some contrasting remarks and/or related results for the case of (1.1) and the LJS-cascade.

As will be seen, two modifications of the original LJS-cascade are made throughout this paper: (i) to permit the phenomena of *stochastic explosion* in cascades where it may naturally happen in unforced equations, and (ii) to exploit time-space self-similarity representation of solutions to (1.1) via a modification of the stochastic cascade. The modified cascade will continue to be referred to as the LJS-cascade.

The organization of this paper is as follows. The main features of a mild formulation of (1.1) based on the LJS-cascade is provided in the next section as motivating background. A few other notable probabilistic approaches to (1.1) are also cited to provide a slightly broader perspective on (1.1), but the primary focus here is the nature of the LJS-cascade in a simpler context of a complex Burgers equation; one may also consult [52] for a survey. As in the case of (1.1), the LJS-cascade dictates a certain probabilistically natural choice for the function spaces in which the Burgers equation can be effectively analyzed. In particular, one is naturally led to Hardy space formulations as defined by functions whose Fourier transform vanish for negative Fourier frequencies and suitably decays for large positive frequencies. For contrast and comparison, the LJS-cascade associated with (1.1) leads naturally to certain Besov-type solution spaces; e.g., see [4, 36]. In any case, once this determination is made, the associated LJS-cascade may be broadly viewed in terms of a time-delayed[2] branching random walk. More specifically, two essential features of the LJS-cascade are (a) the Yule process, viewed as a binary tree-indexed family of i.i.d. mean one exponential random variables, and (b) a corresponding branching random walk in Fourier frequency space.

The LJS-cascade for Burgers leads to the introduction of new related probability models in the forms of a *mean-field Burgers model*, a *β-field Burgers model*, and an *α-delayed Yule process* given in the next two sections. A key feature of the LJS-cascade in general is a certain conservation of *spatial* Fourier frequencies in the corresponding branching random walk. This conservation property persists spatially in the mean-field Burgers model, for which $\beta = \frac{1}{2}$, however is not reflected in the other β-field models. A two-parameter generalization would accommodate the conservation property, however it will not be considered in the present paper beyond a few remarks.

The class of β-field Burgers models conserves temporal frequencies in the case $\beta = \frac{1}{2}$. Due to the difference in scaling of spatial frequencies of a branching random walk and the corresponding scaling of times between movements, one is thus naturally led to a companion purely temporal α-delayed Yule process whose connection to the β-field models can be explicitly expressed via associated Markov semigroups. Temporal frequency conservation is significant among the α-delayed Yule processes and led to the recent discovery that the Poisson process may be realized as an α-delayed Yule process when $\alpha = \frac{1}{2}$, [14]. From a purely probabilistic perspective, this and related results in [14] may be viewed as a variant on an old discovery in [35] identifying the Poisson process as a random time change of a Yule process. It is also noteworthy that $\alpha = \frac{1}{2}$ is a critical value for the α-delayed Yule processes when viewed in terms of boundedness of their infinitesimal generators if and only if $\alpha \leq \frac{1}{2}$. In particular this includes $\alpha = \beta^2 = \frac{1}{4}$ of the mean-field cascade.

Following the brief introduction of these various models, we conclude the paper with a capstone

[2]The mean-field models for the Navier-Stokes equation, on the other hand, involve parameters $\beta > 1$ as well; see [15] in this regard.

section devoted to the analysis of the four leading questions that prompted their consideration: *existence/uniqueness, well-posedness, regularity, and self-similarity.* Although the probabilistic framework is very close to familiar classic models, the perturbations to existing theory provide interesting new challenges for the resolution of these problems.

2 A Brief Highlight of Probabilistic Approaches to Navier-Stokes Equations and the LJS-Cascade

Let us first agree on the signs and normalizations in the Fourier transform to be used throughout. Namely, for integrable functions and/or tempered distributions, suitably interpreted,

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

Two noteworthy observations about (1.1) can be made in terms of Fourier transforms, denoted by $\wedge$:

$$\nabla \cdot v = 0 \quad \text{corresponds to} \quad \xi \cdot \hat{v} = 0$$

and

$$\nabla p \quad \text{corresponds to} \quad \hat{p}\xi.$$

As a result of incompressibility, the pressure term $\hat{p}(\xi, t)$ may be removed while retaining the velocity $\hat{v}(\xi, t)$ by a Leray projection of the (Fourier transformed) equation (1.1) in the direction orthogonal to ξ. As a result of this, the Fourier transformation of multiplication to convolution, and the multiplier effect of the Fourier transform of the Laplacian term, the LJS-cascade emerges naturally as a natural stochastic structure associated with the integrated equations, i.e., a mild form of (1.1). This formulation was generalized in [4], and also developed in [6, 48] from a perspective of harmonic analysis.

Remark 2.1. *The modification (i) of the original LJS-cascade noted above essentially involves the representation of the forcing term. Specifically, in the original LJS-cascade, the unforced equations would be viewed as equations forced by zero, whereas we elect to view the lack of forcing as simply ignorable. This latter view leads to considerations of explosive branching that are not an issue for the original formulation in* [36]*, since a finite explosion time*[3] *means that there will be infinitely many branchings within a finite time.*

To recover p from the projected velocity one notes that, again owing to the incompressibility condition, the divergence of the linear terms in (1.1) is zero. Thus, taking the divergence followed by the Fourier transform, one arrives at

$$\hat{p}(\xi, \cdot) = \sum_{j,k} \hat{R}_j \hat{v}_j(\xi, \cdot) \hat{R}_k \hat{v}_k(\xi, \cdot),$$

[3] An unfortunate typo occurs in the Appendix to [13] in which the explosion event should be denoted $[\zeta < \infty]$, not $[\zeta = \infty]$.

where $\hat{R}_j f(\xi) = -\frac{\xi_j}{|\xi|}\hat{f}(\xi)$ is the Fourier symbol expresses the j-th Riesz transform R_j convolved with f. In particular,

$$p = \sum_{j,k} R_j R_k (v_j v_k).$$

The Gundy-Varapoulos-Silverstein probabilistic representation of Riesz transforms in terms of Brownian motion from infinity is noteworthy in this context. Although originally formulated on a measure space of infinite measure, the construction has been modified in [3] to a measure space with total probability one. To appreciate the role of incompressibility from a probabilistic perspective, one may consider the linearized Stokes problem

$$\frac{\partial v}{\partial t} = \Delta v, \quad \nabla \cdot v = 0,\ x \in \mathbb{R}^3,\ t > 0. \tag{2.1}$$

with arbitrary, not necessarily incompressible, initial data v_0. In [50] the fundamental solution is explicitly computed as

$$v(x,t) = \int_{\mathbb{R}^3} \Gamma(x-z,t) v_0(z) dz, \tag{2.2}$$

where the semigroup $\Gamma = [I_{3\times 3} + \mathcal{R}](K)$, for the 3×3 identity matrix $I_{3\times 3}$, matrix of Riesz transforms $\mathcal{R} = ((R_j R_k))$, and Gaussian transition kernel K for standard Brownian motion. Of course if v_0 is incompressible then the usual representation of solutions to the heat equation in terms of Brownian motion is recovered. A stochastic calculus that would capture the effect of incompressibility on Brownian motion remains an intriguing challenge.

Recently an alternative probabilistic approach to Navier-Stokes was developed by Constantin and Iyer; [32], [33], [11]. Their idea is to use a Weber formula to express the velocity of the inviscid equation in terms of the particle paths, being careful to avoid derivatives in time of the particle paths. For given periodic, incompressible, $2+\delta$-Hölder continuous initial data v_0 this leads to an equivalent system for the inviscid equations Navier-Stokes equations, i.e., Euler equations, of the form

$$\begin{aligned} \dot{X} &= v \\ A &= X^{-1} \\ v &= \mathbf{P}\big[(\nabla^{\text{tr}} A)(v_0 \circ A)\big] \\ X(a,0) &= a, \end{aligned} \tag{2.3}$$

where $\mathbf{P}$ represents the Leray projection onto divergence free vector fields noted earlier, ∇^{tr} is the transpose to the Jacobian, and A_t is the spatial inverse map $A_t(X(t,a)) = a, a \in \mathbb{R}^k (k = 2,3)$. The key idea for the Constantin-Iyer formulation is reflected in their result that, upon replacing the dynamics for the particle trajectories X in (2.3) by a stochastic differential equation $dX = v\,dt + \sqrt{2\nu}\,dW$, the velocity field $v = \mathbb{E}\mathbf{P}\big[(\nabla^{\text{tr}} A)(u_0 \circ A)\big]$ is a fixed point of this modified system if and only if v solves (1.1). In particular, if $\nu = 0$ then this is the system(2.3) for Euler equations. A noteworthy feature of this stochastic framework is that it accommodates domains with boundary conditions beyond periodic [12], while the LJS-cascade theory is restricted to free space and/or periodic boundary conditions by virtue of the Fourier transform.

Earlier probabilistic approaches to (1.1) were introduced in terms of the corresponding vorticity (curl of velocity) equation in [10]; see [24] for a rigorous treatment. Finally, ergodic theory has also provided a natural framework in which (1.1) can be viewed as an infinite dimensional dynamical system; [20–22, 26, 38, 46, 51].

Given the overall strengths and weaknesses of various probabilistic approaches to (1.1), we wish to mention that some practical utility has been demonstrated with the LJS-cascade for computing a convergence rate in a related context of the LANS-α (Lagrange Averaged Navier-Stokes) equations for fluids; a problem posed in [37]. These are essentially the Navier-Stokes equations on a torus except that the spatial scales that are in some sense "smaller than α"are strategically filtered for computational purposes at high Reynolds number. Denoting the solution to LANS-α by $v^{(\alpha)}$, the LJS-cascade may be used to show in suitable function spaces that for $T > 0$,

$$\int_0^T ||v^{(\alpha)}(\cdot, t) - v^{(0)}(\cdot, t)||_{L^2(\mathbf{T})} dt \le A(T)\alpha,$$

for a suitable constant $A(T) > 0$; see [9]. To our knowledge the LJS-cascade has been the only approach to yield a rate in three dimensions, however [7] has subsequently been successful on the two-dimensional problem using more standard pde methods and estimates. In addition to this, the use of LJS-cascades as a numerical Monte Carlo tool has been tested on Burgers equation in [44].

3 Complex Burgers & the LJS-Cascade

The (unforced) viscous Burgers equation is given in free space by

$$\begin{aligned} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} &= \nu \frac{\partial^2 v}{\partial x^2} \\ v(x,0) &= v_0(x) \end{aligned} \tag{3.1}$$

where $v(x, t)$ represents the 1-dimensional velocity at time $t \ge 0$ at position $x \in \mathbb{R}$, $v_0(x)$ represents the initial velocity, and $\nu > 1$ is the viscosity parameter.

The equation above has the following natural symmetry:

If $v(t, x)$ is a solution to Burgers equation, then $v_\lambda(t, x) := \lambda v(\lambda^2 t, \lambda x)$, for $\lambda > 0$, is also a solution corresponding to the initial velocity $(v_0)_\lambda = \lambda v_0(\lambda x)$.

The quantities that are invariant under the above scaling are called *scaling-critical* or *self-similar*. In particular, a solution to (3.1) is called *self-similar* if $v(t, x) = v_\lambda(t, x)$ for all t and x. Of course a self-similar solution would arise from a self-similar initial data $v_0 = (v_0)_\lambda$, and therefore self-similar solutions must be viewed in a function space setting that accommodates this scaling. In Fourier terms this may be expressed in integrated (mild) form as

$$\hat{v}(\xi, t) = \hat{v}_0(\xi)e^{-\nu\xi^2 t} + \frac{i}{2\sqrt{2\pi}}\,\xi \int_0^t e^{-\nu\xi^2(t-s)} \int_{-\infty}^{\infty} \hat{v}(\xi - y, s)\hat{v}(y, s)\, dy\, ds. \tag{3.2}$$

The scaling-symmetry can thusly be expressed in Fourier terms as:

If $\hat{v}(\xi, t)$ is a solution to (3.2), $\hat{v}_\lambda(\xi, t) := \hat{v}(\xi/\lambda, \lambda^2 t)$, for $\lambda > 0$, is also a solution corresponding to the initial velocity $(\hat{v}_0)_\lambda = \hat{v}_0(\xi/\lambda)$.

Consequently self-similar solutions in Fourier space satisfy $\hat{v}(\xi, t) = \hat{v}_\lambda(\xi, t) = \hat{v}(\xi/\lambda, \lambda^2 t)$. In particular this means that self-similar initial data must be piece-wise constant functions of the form:

$$\hat{v}_0(\xi) = \begin{cases} c_1, & \xi > 0; \\ c_2, & \xi < 0\,. \end{cases} \tag{3.3}$$

This means that in order to accommodate the self-similar case, one must consider settings that include initial data in L^∞ in Fourier space.

For the LJS-cascade we seek to represent the mild formulation (3.2) in terms of expected values of products of initial data along paths of an evolving random binary tree. The exponential density in t is evident, and the space integral naturally occurs as an expected value under the above-mentioned idea of self-similarity if we set $\hat{v}_0(\xi) = 0$ for $\xi < 0$ ($c_2 = 0$ in (3.3)). For then $\hat{v}(\xi, t) = 0$ for all $\xi < 0$ and $t > 0$, and the mild formulation (3.2) becomes

$$\hat{v}(\xi,t) = \hat{v}_0(\xi)e^{-\nu\xi^2 t} + \frac{i}{2\sqrt{2\pi\nu}} \int_0^t \nu\xi^2 e^{-\nu\xi^2\tau} \frac{1}{\xi} \int_0^\xi \hat{v}(\eta, t-\tau)\hat{v}(\xi-\eta, t-\tau)d\eta d\tau, \quad \xi > 0,\ t \geq 0. \tag{3.4}$$

Note that the convolution integral in the above formulation is an integral with respect to a probability distribution concentrated on $\{(\eta_1, \eta_2) \in (0, \xi) \times (0, \xi) : \eta_1 + \eta_2 = \xi\}$ having uniformly distributed marginals on $(0, \xi)$, respectively. These can, in turn, be rescaled in terms of uniform distributions on $(0, 1)$. The linear relation between Fourier frequencies (or wave numbers) is referred to as (spatial) *conservation of frequencies*. Of course the implied asymmetry of the Fourier transform necessitates consideration of complex-valued solutions v.

The natural function space settings that is associated with (3.4) is that of a Hardy-type space

$$\mathcal{H}_\infty = \{v \in \mathcal{D}'(\mathbb{R} : \mathbb{C}) :\ \hat{v}(\xi) = 0 \text{ for } \xi < 0, \hat{v} \in L^\infty([0, \infty), \mathbb{C}))\} \tag{3.5}$$

Also, we set

$$\|v\|_{\mathcal{H}_\infty} = \inf\{M \geq 0 :\ |\hat{v}(\xi)| \leq M \text{ a.e. } \xi > 0\} \tag{3.6}$$

Remark 3.1. In fact, one can show that the real self-similar solutions to (3.1) must be of the form

$$v(x, t) = \begin{cases} \frac{c_1}{x}, & x > 0; \\ \frac{c_2}{x}, & x < 0\,. \end{cases} \tag{3.7}$$

where $c_1, c_2 \in \{0, -2\}$. Therefore, to obtain a nontrivial LJS-cascade theory that includes the self-similar case, one must consider complex-valued solutions v.

Note that upon rescaling ξ by $\frac{1}{\sqrt{\nu}}\xi$ and multiplying the equation by $\frac{i}{2\sqrt{2\pi\nu}}$; i.e., a dilation-rotation in the complex plane, the factor $\frac{i}{2\sqrt{2\pi\nu}}$ of the integral term is removed and $\nu = 1$ in (3.4) so transformed. Therefore, to simplify the notation we will adopt the following convention for the rest of the paper.

Convention: *From here out, unless otherwise stated, we rescale and assume the convention for a rotation-dilation of the complex plane to both render $\nu = 1$, and to remove the factor $\frac{i}{2\sqrt{2\pi\nu}}$. For notational simplicity, we continue to denote the transformed function by $\hat{v}$.* (C)

Using this Convention, (3.4) becomes

$$\hat{v}(\xi,t)=\hat{v}_0(\xi)e^{-\xi^2 t}+\int_0^t \xi^2 e^{-\xi^2\tau}\frac{1}{\xi}\int_0^\xi \hat{v}(\eta,t-\tau)\hat{v}(\xi-\eta,t-\tau)d\eta d\tau,\quad \xi>0,\ t\geq 0. \quad (3.8)$$

This is the mild formulation of the Fourier transformed complex Burgers equation that will be of main focus in this paper. In addition, related “mean field”equations will also be introduced in due course.

The LJS-cascade corresponding to (3.8), to be referred to as the Burgers cascade, consists of (i) a Yule process

$$Y=\{T_s : s\in\mathbb{T}=\cup_{n=0}^\infty\{1,2\}^n\} \quad (3.9)$$

defined by a collection of i.i.d. mean one exponentially distributed random variables T_s indexed by vertices s of the full binary tree $\mathbb{T}=\cup_{m=0}^\infty\{1,2\}^m$, $\{1,2\}^0=\{\theta\}$; (ii) a multiplicative branching random walk starting from $\xi\neq 0$, recursively defined by

$$W_\theta(\xi)=\xi,\quad W_s(\xi)=U_s W_{s|m-1}(\xi),\quad s\in\mathbb{T},\ |s|=m\geq 1, \quad (3.10)$$

where $\{(U_{s1},U_{s2}) : s\in\mathbb{T}\}$ is a collection of i.i.d. random vectors, independent of Y, having uniformly distributed marginals on $(0,1)$, and satisfying the conservation of frequency constraint

$$U_{s1}+U_{s2}=1,\quad s\in\mathbb{T}. \quad (3.11)$$

Here we use $|\theta|=0, |s|=|(s_1,\dots,s_n)|=n$ to denote the generation of $s\in\mathbb{T}$, and for $|s|\geq n$, $s|0=\theta, s|j=(s_1,\dots,s_j), j\geq 1$, denotes $s\in\mathbb{T}$ restricted to the first j generations. In the special case $\xi=1$ we will simply write W in place of $W(1)$.

The following stochastic processes defining the cascade genealogy are convenient for the analysis of the Burgers cascade.

Definition 3.1. *The genealogy of the complex Burgers cascade is the set-valued branching stochastic process*

$$V_{\mathrm{unif}}(\xi,t)\equiv\{s\in\mathbb{T} : \sum_{j=0}^{|s|-1}W_{s|j}^{-2}T_{s|j}\leq\xi^2 t<\sum_{j=0}^{|s|}W_{s|j}^{-2}T_{s|j}\}\in\mathcal{E}, \quad (3.12)$$

where $\mathcal{E}$ is the space of evolutionary sets $V\subset\mathbb{T}$ inductively defined by $V=\{\theta\}$ and, for $\#V\geq 2$, $V=W\backslash s\cup\{s1,s2\}$ for some $W\in\mathcal{E}$, $\#W=\#V-1$, $s\in W$, where $\#A$ denotes the cardinality of a set A. The number of cascade leaves at time t is denoted

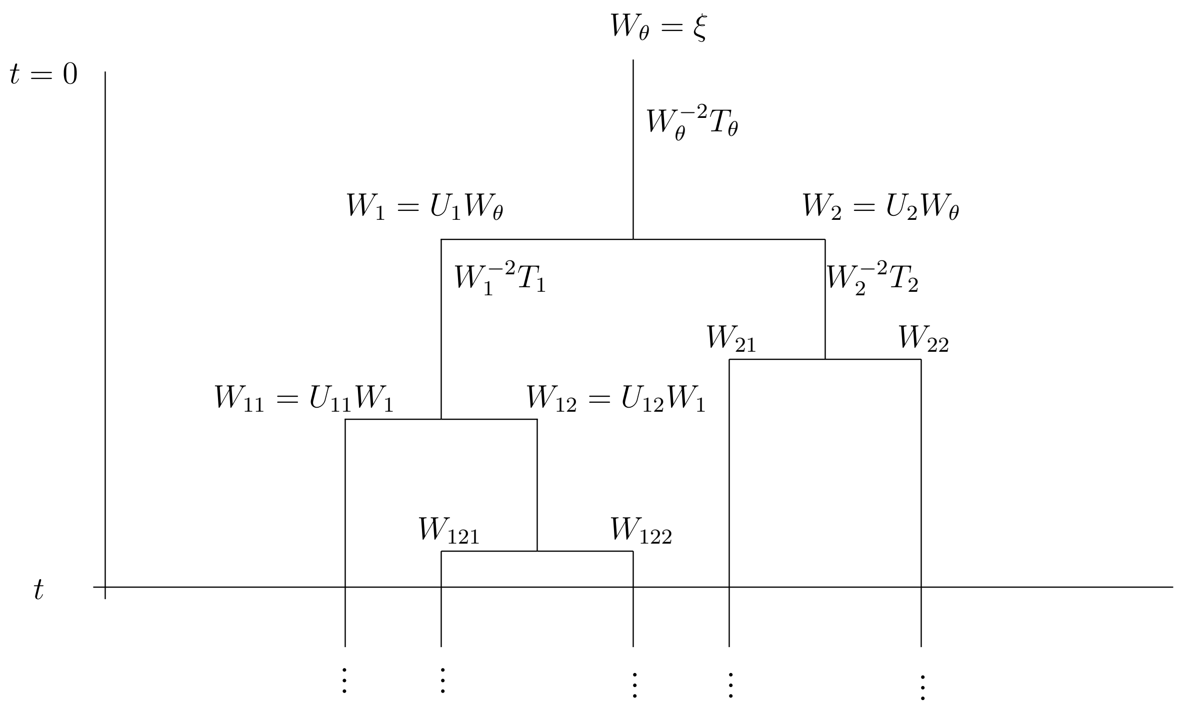
$$N_{\mathrm{unif}}(\xi,t)=\#V_{\mathrm{unif}}(\xi,t),\quad t\geq 0,\xi>0. \quad (3.13)$$

The (stochastic) explosion time for the complex Burgers cascade is the non-negative extended real-valued random variable

$$\zeta=\lim_{n\to\infty}\min_{|s|=n}\sum_{j=1}^n W_{s|j}^{-2}T_{s|j}. \quad (3.14)$$

The event that explosion occurs is defined by $[\zeta<\infty]$.

Note that the state space $\mathcal{E}$ of evolutionary sets is a denumerable set, and $V_{\mathrm{unif}}(\xi,t)$ may be viewed as a random binary subtree of $\mathbb{T}$ rooted at θ.



Remark 3.2. This is essentially the LJS-cascade modified for the absence of forcing and applied to Burgers equation. If the explosion time is finite then the number of branchings within a finite time horizon will be infinite. In such cases the recursive definition below (3.15) of the cascade must also be modified. In the case of (1.1), one makes the orthogonal projection noted earlier to remove the pressure term prior to taking Fourier transforms. The convolution term remains as an integral over $\mathbb{R}^3$ with a variety of normalizations (majorizations) available, see [4, 36], to obtain an integral with respect to a probability distribution of conserved wave number pairs. This latter flexibility provides alternative function spaces for solutions.

Based on the LJS-cascade described above, a "stochastic solution" may be expressed iteratively as:

$$X(\xi,t) = \prod_{s \in V_{\text{unif}}(\xi,t)} \hat{v}_0(W_s(\xi)) = \begin{cases} \hat{v}_0(\xi), & T_\theta \geq t \\ X^{(1)}(W_1(\xi), t - T_\theta) X^{(2)}(W_2(\xi)), t - T_\theta), & T_\theta < t, \end{cases} \tag{3.15}$$

where $X^{(1)}, X^{(2)}$ are independent copies of X.

The LJS-cascade solution to (3.8) is obtained from the stochastic solution (3.15) via

$$\hat{v}(\xi,t) = \mathbb{E}X(\xi,t) = \mathbb{E} \prod_{s \in V_{\text{unif}}} \hat{v}_0(W_s). \tag{3.16}$$

Clearly, if the expected value above is well-defined, then the LJS-cascade solution (3.16) solves (3.8).

In Section 6 we will consider the existence of the expected value in (3.16) as well as the following basic questions driving the problems, conjectures and partial results pertaining to (3.8) and related models.

(Q1) Existence/Uniqueness of Mild Solutions: *A probabilistic rendering of (3.8) naturally suggests analysis in a Hardy-type space $\mathcal{H}_\infty$. The questions of existence and uniqueness may be analyzed globally in time for a subset of initial data, or locally in time for all initial data.*

(Q2) Global Well-Posedness (frequency asymptotic): *This question pertains to the identification of linear subspaces of $\mathcal{H}_\infty$ for which mild solutions exist for all time and remain in the subspace. Regularity and/or self-similarity considerations, i.e., Q3, Q4, may impose specific conditions on solution spaces for Q1 and Q2.*

(Q3) Regularity: *The positive form of this question involves the C^∞ behavior of solutions for all time for smooth (Schwarz space) initial data. This question includes finite time unbounded growth of the Fourier transform.*

(Q4) Self-Similarity: *The positive form of this question involves the uniqueness of solutions under conditions of unique space/time scale-invariant solutions. The absence of such uniqueness defines a notion of symmetry breaking. The solution space is naturally required to contain constants since these provide self-similar solutions.*

We will see that (Q1) can be answered in the affirmative. In particular, for Burgers equation and the related LJS-cascade we will prove the following basic result (see Theorem 6.1).

Theorem 3.1. *For $\xi \neq 0$, with probability one*

- *No Stochastic Explosion:* $\quad \zeta = \infty.$
- *∞-Radius of Convergence:* $\quad \sup\{r \geq 1 : \mathbb{E}r^{N(\xi,t)} < \infty\} = \infty, \quad t \geq 0.$

Moreover, for $v_0 \in \mathcal{H}_\infty$

$$\hat{v}(\xi, t) = \mathbb{E} \prod_{s \in V_{\text{unif}}(\xi,t)} \hat{v}_0(W_s(\xi)), \quad \xi > 0, t \geq 0,$$

is the unique mild solution to (3.8).

On the other hand (Q2) and (Q3) are delicate even in this substantially reduced framework of complex Burgers. Their resolution can take interesting twists and turns, suggesting still further questions about the nature of quasilinear equations and their companion delayed branching random walks.

4 Mean-field Burgers & β-field Burgers

The mean-field Burgers model is defined under Convention C by replacing U by the constant $\beta = \mathbb{E}U = \frac{1}{2}$ in the LJS-cascade for complex Burgers (3.8) defined in the previous section. More generally, the β-field Burgers model is defined by replacing U by a specified constant $\beta \in [0, 1]$ for spatial frequencies, i.e., $W_s(\xi) = \beta^{|s|}\xi$ defines the (multiplicative) branching random walk. As a result $W_s(\xi)^2$ is replaced by $W_s(\xi)^2 = \beta^{2|s|}\xi^2 \in [0, 1], s \in \mathbb{T}$, when scaling the temporal frequencies. To keep the notation simple, we let the context indicate the meaning of W, continuing to suppress ξ when $\xi = 1$. Now define

$$V_\beta(\xi,t) = \{s \in \mathbb{T} : \sum_{j=0}^{|s|-1} \beta^{-2j} T_{s|j} \le \xi^2 t < \sum_{j=0}^{|s|-1} \beta^{-2j} T_{s|j}\}, \quad t \ge 0. \tag{4.1}$$

Then the explosion time is accordingly replaced by

$$\zeta_\beta = \lim_{n\to\infty} \min_{|s|=n} \sum_{j=0}^{n} \beta^{-2j} T_{s|j}. \tag{4.2}$$

The mean-field Burgers cascade is the β-field Burgers cascade with $\beta = 1/2$. Observe that the β-field Burgers cascade has the associated mild equation obtained by replacing the uniform distribution by the Dirac distribution $\delta_{\{\beta\}}$:

$$\hat{v}(\xi,t) = \hat{v}_0(\xi) e^{-\xi^2 t} + \int_0^t \xi^2 e^{-\xi^2\tau} \hat{v}^2(\beta\xi, t-\tau) d\tau, \quad \xi \ne 0,\ t \ge 0. \tag{4.3}$$

The corresponding pde associated with the mild β-field cascade is given by

$$\frac{\partial \hat{v}(\xi,t)}{\partial t} = -\xi^2 \hat{v}(\xi,t) + \xi^2 \hat{v}^2(\beta\xi,t),\ t > 0, \quad \hat{v}(\xi,0) = \hat{v}_0(\xi). \tag{4.4}$$

Remark 4.1. Although it will not be considered beyond the mean-field Burgers cascade in this paper, it is also natural to consider a two-parameter *mixed* (β_1,β_2)*-field cascade* where $0 \le \beta_1, \beta_2 \le 1$, and $\beta_1 + \beta_2 = 1$, corresponding to the equation

$$\hat{v}(\xi,t) = \hat{v}_0(\xi) e^{-\xi^2 t} + \int_0^t \xi^2 e^{-\xi^2\tau} \hat{v}(\beta_1\xi, t-\tau)\hat{v}(\beta_2\xi, t-\tau) d\tau, \quad \xi \ne 0,\ t \ge 0. \tag{4.5}$$

In cases of the two extremes $\beta = 0$ and $\beta = 1$, one has the following explicit solutions:

$$\beta = 0: \qquad \hat{v}(\xi,t) = e^{-\xi^2 t}\hat{v}_0(\xi) + (1 - e^{-\xi^2 t})\hat{v}_0^2(0), \quad t \ge 0, \tag{4.6}$$

and, noting for the Yule process that $N_1 = \#V_1(\xi,t)$ has a geometric distribution with parameter $p_t = e^{-\xi^2 t}$, or simply solving (4.4) in the case $\beta = 1$,

$$\beta = 1: \qquad \hat{v}(\xi,t) \equiv \mathbb{E}\hat{v}_0(\xi)^{\#V_1(\xi,t)-1} = \frac{\hat{v}_0(\xi)e^{-\xi^2 t}}{1 - \hat{v}_0(\xi) + \hat{v}_0(\xi)e^{-\xi^2 t}}, \quad 0 \le t < t_\infty(\hat{v}_0(\xi)), \tag{4.7}$$

respectively, where

$$t_\infty(\hat{v}_0(\xi)) = \begin{cases} \frac{1}{\xi^2} \ln\left(\frac{\hat{v}_0(\xi)}{\hat{v}_0(\xi)-1}\right), & \text{for } \ \hat{v}_0(\xi) > 1 \\ \infty, & \text{for } \ -\infty < \hat{v}_0(\xi) \le 1. \end{cases} \tag{4.8}$$

In particular, while the solution in the case $\beta = 0$ preserves the structure of the initial data over all time, in the case $\beta = 1$ there is finite-time blow-up for any initial data with $\hat{v}_0(\xi) > 1$. In fact, if $\hat{v}_0(\xi) = M > 1, \xi > 0$ is constant then $\hat{v}$ instantaneously exits $\mathcal{H}_\infty$, i.e., $\inf_{\xi>0} t_\infty(\hat{v}_0(\xi)) = 0$, making the problem ill-posed in $\mathcal{H}_\infty$. Even if $\hat{v} = M\mathbf{1}_{[a,b]}, M > 1$, is compactly supported on $0 < a < b < \infty$, there is finite time blow-up at

$$t_\infty(\hat{v}_0) = \inf_{\xi>0} t_\infty(\hat{v}_0(\xi)) = b^{-2}\frac{\ln M}{\ln(M-1)}.$$

The mean field value $\beta = \frac{1}{2}$ is of particular interest, but as shown in the next section, the parameter $\beta = \frac{1}{\sqrt{2}}$ also results in distinguished structure.

5 α-Delayed Yule Process

The α-delayed Yule process is defined by the β-field Burgers cascade under Convention C with $\alpha = \beta^2, \xi = 1$. With this reduction in parameters, for $\alpha \in (0,1]$, the α-delayed Yule process is simply denoted

$$V^{(\alpha)}(t) = V_\beta(\xi, t),\ (\beta = \sqrt{\alpha},\ \xi = 1), \quad t \geq 0. \tag{5.1}$$

The mean-field model corresponds to $\alpha = \frac{1}{4} < \frac{1}{2}$. The Yule process is then obtained in this context when $\alpha = 1 > \frac{1}{2}$. The parameter value $\alpha = \frac{1}{2}$ may be viewed as a critical value of the β-field evolutions as explained below. Moreover, as also shown in [14] the $\frac{1}{2}$-delayed Yule process is the (shifted) Poisson process with unit intensity; in fact, this extends to a two-parameter delayed Yule process provided $\alpha_1 + \alpha_2 = 1$.

Give $\mathcal{E}$ the discrete topology and let $C_0(\mathcal{E})$ denote the space of (continuous) real-valued functions $f : \mathcal{E} \to \mathbb{R}$ that vanish at infinity; i.e., given $\epsilon > 0$, one has $|f(V)| < \epsilon$ for all but finitely many $V \in \mathcal{E}$, with the uniform norm $||\cdot||_u$. The subspace $C_{00}(\mathcal{E}) \subset C_0(\mathcal{E})$ of functions with compact (finite) support is clearly dense in $C_0(\mathcal{E})$ for the uniform norm.

Since for each $0 < \alpha \leq 1$, (5.1) defines a Markov process[4] $V^{(\alpha)}$, one has corresponding semigroups of positive linear contractions $\{T_t^{(\alpha)} : t \geq 0\}$ defined by

$$T_t^{(\alpha)} f(V) = \mathbb{E}_V f(V^{(\alpha)}(t)), \quad t \geq 0, f \in C_0(\mathcal{E}), \tag{5.2}$$

with the usual branching process convention that given $V^{(\alpha)}(0) = V \in \mathcal{E}$, $V^{(\alpha)}(t)$ is the union of those progeny at time t independently produced by single progenitors at each node $s \in V$.

The connection with the β-field model under Convention C and for suitable $\hat{v}_0$ may be expressed in terms of the semigroups as

$$\hat{v}(\xi, t) = T_{\beta^2 t}^{(\beta^2)} \varphi(\hat{v}_0, \xi, \beta; \cdot)(\emptyset), \tag{5.3}$$

where, for real $\hat{v}_0$ and Convention C, $\varphi(\hat{v}_0, \xi, \beta; \cdot) : \mathcal{E} \to \mathbb{R}$ is given by

$$\varphi(\hat{v}_0, \xi, \beta; V) = \prod_{s \in V} \hat{v}_0(|s|^\beta \xi), \quad V \in \mathcal{E}. \tag{5.4}$$

The usual considerations imply that the infinitesimal generator $(A^{(\alpha)}, \mathcal{D}_\alpha)$ of $V^{(\alpha)}$ is given on $C_{00}(\mathcal{E})$ via

$$A^{(\alpha)} f(V) = \sum_{s \in V} \alpha^{|s|}\{f(V^s) - f(V)\}, \quad f \in C_{00}(\mathcal{E}), \tag{5.5}$$

[4]Another closely related Markov evolution that takes place in the sequence space ℓ_1 is given in [14].

where

$$V^s = V\backslash\{s\} \cup \{< s1, s2 >\}, \quad s \in V.$$

Proposition 5.1. *The space $C_{00}(\mathcal{E})$ of continuous functions on $\mathcal{E}$ having compact (finite) support is a core for the generator of the semigroup defined by (5.2). In particular one has*

$$\frac{\partial}{\partial t} T_t^{(\alpha)} f(V) = A^{(\alpha)} T_t^{(\alpha)} f(V) = T_t^{(\alpha)} A^{(\alpha)} f(V),\ t > 0, \quad f \in \mathcal{D}_\alpha \supset C_{00}(\mathcal{E}).$$

Proof. Note that $f \in C_{00}(\mathcal{E})$ if and only if there are $W_1, \ldots, W_m$ in $\mathcal{E}$, and real numbers $f_1, \ldots, f_m$ such that $f = \sum_{j=1}^m f_j \delta_{\{W_j\}}$. Thus, if $\#V > \#W$, then

$$T_t^{(\alpha)} f(V) = \sum_{j=1}^m f_j P(V^{(\alpha)}(t) = W_j | V^{(\alpha)}(0) = V) = 0.$$

In particular, $T_t^{(\alpha)} f \in C_{00}(\mathcal{E})$. Since $C_{00}(\mathcal{E})$ is dense in $C_0(\mathcal{E})$, the assertion follows from standard semigroup theory. □

The following result from [14] displays a distinct role of $\alpha = \frac{1}{2}$ as a critical parameter in terms of boundedness of the infinitesimal generators.

Proposition 5.2. $(A^{(\alpha)}, \mathcal{D}_\alpha)$*,* $\mathcal{D}_\alpha \subset C_0(\mathcal{E})$ *is a bounded linear operator if and only if* $\alpha \leq \frac{1}{2}$*. In particular, the infinitesimal generator is a bounded operator for the mean-field parameter* $\alpha = \frac{1}{4}$*.*

Other interesting values of $\alpha \in (0, 1)$ arise by consideration of

$$m^{(\alpha)}(t) = \mathbb{E}N^{(\alpha)}(t).$$

Proposition 5.3. *For* $0 < \alpha \leq 1$*,*

$$\mathbb{E}N^{(\alpha)}(t) = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} (2\alpha^j - 1) \frac{t^n}{n!}, \quad t \geq 0. \tag{5.6}$$

In particular, $t \to \mathbb{E}N^{(\alpha)}(t)$ *is a polynomial in* t *of degree* k *for any* α *that is a* k*-th root of* $\frac{1}{2}$ *for some* $k = 1, 2, \ldots$*.*

Proof. One may readily observe, e.g, by conditioning on T_θ, that

$$\frac{dm^{(\alpha)}}{dt} = -m^{(\alpha)}(t) + 2m^{(\alpha)}(\alpha t),\ m^{(\alpha)}(0) = 1. \tag{5.7}$$

From here one may either derive the asserted formula by series expansion, or check the assertion directly. The polynomial solutions are made obvious by inspection. □

Remark 5.1. The positive functions

$$a_\beta(V) = \sum_{s \in V} \beta^{|s|}, \quad V \in \mathcal{E}, \tag{5.8}$$

provide a class of genealogical gauges on evolutionary sets. In particular, under the convention $0^0 = 1$, $a_0(V) = \delta_{\{\theta\}}(V)$, and $a_1(V) = \#V, V \in \mathcal{E}$. Although $a_\beta \notin C_0(\mathcal{E})$ for any $\beta \in (0, 1]$, the following formal calculation for $\alpha \in (0, 1]$,

$$A^{(\alpha)} a_\beta(V) = (2\beta - 1)a_{\alpha\beta}(V), \quad V \in \mathcal{E},$$

leads to a class of positive martingales associated with the Yule process given by

$$M(t) = e^{(2\beta-1)t} a_\beta(V^{(1)}(t)), \quad t \geq 0,$$

which is shown in [14] to be uniformly integrable if and only if $\beta < \beta_c$, where $\beta_c \approx 0.1867$ is the unique solution in $(0, 1]$ to

$$\beta_c \ln \beta_c = \beta_c - 1.$$

The associated semigroup equations become available by the following. Define a sequence $a_\beta^{(n)} \in C_{00}(\mathcal{E})$ by restricting the positive support of a_β to $\mathcal{E}^{(n)} = \{V \in \mathcal{E} : \#V \leq n\}$, for $n = 1, 2 \ldots,$ respectively. Then, one has $\mathcal{E} = \cup_{n=1}^{\infty} \mathcal{E}^{(n)}, \mathcal{E}^{(1)} \subset \mathcal{E}^{(2)} \subset \cdots$, and

$$A^{(\alpha)} a_\beta^{(n)}(V) = (2\beta - 1)a_{\alpha\beta}(V), \quad V \in \mathcal{E}^{(n-1)}, n = 2, 3, \ldots.$$

6 Basic Problems for Complex Burgers & Mean-field Burgers: Some results and conjectures

We will start with several general results about probabilistic properties of LJS-cascades for Burgers and β-field Burgers equations. In the subsequent subsections we will use these results to analyze well-posedness and regularity issues for the solutions to the corresponding PDE.

As a matter of notation, the evolution sets $V_\bullet(\xi, t)$ as well as generic functionals, e.g, $N_\bullet(\xi, t)$, will be denoted without specific subscripts: i.e., V will be used for either V_{unif} or V_β and similarly, $N(\xi, t) = \#V(\xi, t)$ will stand for either N_{unif} or N_β, as dictated by context.

First, we note that the tree structures of the corresponding LJS-cascades are independent on the initial data in (3.8) or (4.3), and thus must preserve the scaling-invariance $(\xi, t) \to (\lambda^{-1}\xi, \lambda^2 t)$. As a result, the following self-similarity and monotonicity properties are straightforward to prove and are useful in some of the general analysis:

$$\mathbf{Self-Similarity}: \quad \begin{array}{l} V(\lambda^{-1}\xi, \lambda^2 t) = V(\xi, t) \equiv V(\tau) \\ N(\lambda^{-1}\xi, \lambda^2 t) = N(\xi, t) \equiv N(\tau) \end{array} \quad \lambda > 0, \ \tau = \xi^2 t. \tag{6.1}$$

Definition 6.1. *For evolutionary sets $W, V \in \mathcal{E}$ we say V precedes W, denoted $V \prec W$, if each $s \in V$ has a (possibly empty) concatenation belonging to W, i.e., for each $s \in V$ either $s \in W$ or there is an $\overline{s} \in \{1, 2\}^m$, for some $m \geq 1$, such that $s * \overline{s} \in W$.*

It is staight-forward to check that

$$V^{(\alpha_1)}(t) \prec V^{(\alpha_2)}(t), \quad 0 < \alpha_2 \leq \alpha_1 \leq 1, \ \xi, t \geq 0. \tag{6.2}$$

$$V_{(\beta_1)}(\xi, t) \prec V_{(\beta_2)}(\xi, t), \quad 0 < \beta_2 \leq \beta_1 \leq 1, \ \xi, t > 0. \tag{6.3}$$

$$V(\tau_1) \prec V(\tau_2), \quad 0 < \tau_1 < \tau_2. \tag{6.4}$$

As a consequence one has the following.

Proposition 6.1. *For a fixed $\tau > 0$, the following functionals are increasing in $\alpha = \beta^2$.*

$$N_\beta = H^{(\alpha)}(\tau) = \# V_\beta(\tau) \quad \textit{and} \quad H_\beta(\tau) = H^{(\alpha)}(\tau) = \max_{s \in V^{(\alpha)}(\tau)} \sum_{j=0}^{|s|-1} (\frac{1}{\alpha})^j T_{s|j},$$

The next property, specifically the *non-explosion* of the LJS-cascades, is crucial for the analysis in the subsections to follow. The non-explosion becomes obvious if we compare Burgers and β-field cascades to the Yule process.

Proposition 6.2. *Let $\tau = \xi^2 t$, $\beta \in [0, 1]$. Then*

$$P(N_\beta(\tau) < \infty) = P(N_{\text{unif}}(\tau) < \infty) = 1\,,$$

and consequently, Burgers and β-Burgers cascades are non-exploding.

Proof. First, we note that $P(N_1(\tau) < \infty) = 1$ for the Yule process, ($\alpha = \beta = 1$) since, as is well-known and easily checked, $N_1(\tau)$ is distributed geometrically: $P(N_1(\tau) = n) = e^{-\tau}(1 - e^{-\tau})^{n-1}$, and therefore,

$$P(N_1(\tau < \infty) = \sum_{n=1}^{\infty} P(N_1(\tau) = n) = 1.$$

Now observe that since for Burgers and β-field trees, $W_s(\xi) \leq \xi$, $N_{\text{unif}}(\tau) \leq N_1(\tau)$ and $N_\beta \leq N_1(\tau)$ a.s., and so $P(N_{\text{unif}}(\tau) < \infty) = P(N_\beta(\tau) < \infty) = 1$. The non-explosion immediately follows. □

The following estimate on the distribution of N will be used in Subsection 6.1 to establish finiteness of the expected value (3.16).

Proposition 6.3. *Let $\tau = \xi^2 t$ and denote $P_k(\tau) := P(N_{\text{unif}}(\tau) = k)$, we have*

$$P_k(\tau) \leq e^{-\tau/k} \frac{\tau^{k-1}}{(k-1)!}\,, \qquad k \in \mathbb{N}\,. \tag{6.5}$$

Proof. The estimate (6.5) is proved by induction. For $n = 1$ we have $P_1(\tau) = P(T_\theta < t) = e^{-\tau}$. Assume (6.5) holds for $k \leq n$. Then, conditioning on the first branching,

$$\begin{aligned} P_{n+1}(\tau) &= \int_0^t \xi^2 e^{-\xi^2 s} \frac{1}{\xi} \int_0^\xi \sum_{k=1}^n P_k(y^2 t - s) P_{n+1-k}((\xi - y)^2 (t - s))\, dy\, ds \\ &= e^{-\tau} \int_0^\tau e^\sigma \int_0^1 \sum_{k=1}^n P_k(\eta^2 \sigma) P_{n+1-k}((1 - \eta)^2 \sigma)\, d\eta\, d\sigma \\ &\leq e^{-\tau} \int_0^\tau e^\sigma \int_0^1 \sum_{j=0}^{n-1} e^{-\left(\frac{\eta^2}{k} + \frac{1-\eta^2}{n-k}\right)\sigma} \frac{(\eta^2 \sigma)^{k-1}}{(k-1)!} \frac{((1-\eta)^2 \sigma)^{n-k}}{(n-k)!}\, d\eta\, d\sigma. \end{aligned}$$

We will use the following lemma.

Lemma 6.1. *For any $\eta \in [0,1]$, $n \in \mathbb{N}$, and $k \in \{1,\dots n\}$, one has:*

$$\frac{1}{n+1} \le \frac{\eta^2}{k} + \frac{(1-\eta)^2}{n+1-k} \quad \left(\le \max\left\{\frac{1}{k}, \frac{1}{n+1-k}\right\}\right) \tag{6.6}$$

Proof. The lemma follows by considering the extrema of the function $\phi(\eta) = \frac{\eta^2}{k} + \frac{(1-\eta)^2}{n+1-k}$ on $\eta \in [0,1]$. □

Using (6.6) we obtain

$$\begin{aligned} P_{n+1}(\tau) &\le e^{-\tau} \int_0^\tau e^{\frac{n}{n+1}\sigma} \frac{1}{(n-1)!} \int_0^1 \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} (\eta^2\sigma)^j ((1-\eta)^2\sigma)^{n-1-j}\, d\eta\, d\sigma \\ &= \frac{e^{-\tau}}{(n-1)!} \int_0^\tau e^{\frac{n}{n+1}\sigma} \int_0^1 \left((\eta^2 + (1-\eta)^2)\sigma\right)^{n-1}\, d\eta\, d\sigma \\ &\le \frac{e^{-\tau}}{(n-1)!} \left(\frac{n+1}{n}\right)^n \int_0^\tau e^{\frac{n}{n+1}\sigma} \left(\frac{n}{n+1}\sigma\right)^{n-1} d\left(\frac{n}{n+1}\sigma\right), \end{aligned}$$

where we used $\eta^2 + (1-\eta)^2 \le 1$ on $\eta \in [0,1]$.

To further estimate the integral above we note that by the mean value theorem,

$$\int_0^x e^y y^{n-1}\, dy \le \frac{x^n}{n} e^x \tag{6.7}$$

for $x \ge 0$ and $n \in \mathbb{N}$.

Thus, using the (6.7) with $y = \frac{n}{n+1}\sigma$, we obtain:

$$P_{n+1}(\tau) \le \frac{e^{-\tau}}{(n-1)!} \left(\frac{n+1}{n}\right)^n \frac{\left(\frac{n}{n+1}\tau\right)^n}{n} e^{\frac{n}{n+1}\tau} = e^{-\frac{\tau}{n+1}} \frac{\tau^n}{n!},$$

and so (6.5) holds for $k = n+1$.

□

In the case β-field models one has the following bounds.

Proposition 6.4. *Let $\tau = \xi^2 t$ and $0 < \beta^2 = \alpha \le 1$ and denote $P_k^{(\alpha)}(\tau) := P(N_\beta(\tau) = k)$. Then for $k \ge 2$ we have:*

$$P_k^{(\alpha)}(\tau) \le \frac{e^{-(2\beta^2)^{k-2}\tau}}{2\beta^2} \frac{(2\beta^2\tau)^{k-1}}{(k-1)!}\,, \qquad \alpha = \beta^2 < \frac{1}{2}; \tag{6.8}$$

$$P_k^{(1/2)}(\tau) = e^{-\tau} \frac{\tau^{k-1}}{(k-1)!}\,, \qquad \alpha = \beta^2 = \frac{1}{2}. \tag{6.9}$$

and

$$P_k^{(\alpha)}(\tau) \le e^{-\tau} \frac{\left(1 - e^{-(2\beta^2-1)\tau}\right)^{k-1}}{(2\beta^2-1)^{k-1}}\,, \qquad \alpha = \beta^2 > \frac{1}{2}; \tag{6.10}$$

Proof. The statements can be proved by induction analogously to (6.5). Note that in the β-field case, we have for $\alpha = \beta^2$

$$P_{n+1}^{(\alpha)}(\tau) = e^{-\tau}\int_0^\tau e^\sigma \sum_{k=1}^n P_k^{(\alpha)}(\alpha\sigma)P_{n+1-k}^{(\alpha)}(\alpha\sigma)\,d\sigma\,,$$

and so, for $\beta = 1/\sqrt{2}$ the exponentials inside the integral disappear, leading to Poisson distribution (6.9).

In the case $\beta^2 \in (1/2, 1]$ we use bound $(1 - e^{-(2\beta^2-1)\beta^2\sigma}) \le (1 - e^{-(2\beta^2-1)\sigma})$ in the inductive step.

□

Remark 6.1. We note that by Proposition 6.1 the tails of N_β become thinner as $\beta \to 0$.

The next result will be relevant to establish lack of well-posedness in Hadamard sense.

Proposition 6.5. *Let $\tau = \xi^2 t$, $\beta \in [0, 1]$. Then, as $\tau \to \infty$*

$$\mathbb{E}\, r^{N_{\text{unif}}(\tau)},\ \mathbb{E}\, r^{N_\beta(\tau)} \to \begin{cases} 0, & 0 \le r < 1 \\ 1, & r = 1 \\ \infty & r > 1. \end{cases}$$

Proof. We note that the numbers of nodes $N_\bullet(\tau) \nearrow \infty$ as $\tau \to \infty$. Then the assertion follows from the monotonicity in τ of the sequence $r^{N_\bullet(\tau)}$ by applying the monotone convergence theorem for $r \ge 1$ and the dominated convergence theorem for $0 \le r < 1$. □

For the analysis of regularity, it is important to obtain estimates on $L(\xi, t)$, and $R(\xi, t)$ – left-most and right-most (delayed) branching random walkers. In particular, the properties below will prove sufficient to capture initial data in $\mathcal{H}_\infty$ with compact support in ξ which will be analyzed in Subsection 6.3.

Remark 6.2. A rather complete probabilistic analysis of branching random walks associated with Yule processes ($\beta = 1$) has been evolving in the probability literature over the past several decades. The recent paper [45] is especially appropriate to the present setting in its focus on the *heights, fully saturated trees,* and the *saturation height*; also see [1, 5, 8, 17, 25, 27, 34, 43] for related results of various types. However the corresponding problems for the Burgers cascade and/or the β-field cascades involve temporal delays to the Yule structure that make the analysis of the relevant functionals more challenging in cases other than $\beta = 1$.

In view of the results of the previous section on the α-delayed Yule process, the case $\alpha = \frac{1}{2}$, or $\beta = \frac{1}{\sqrt{2}}$, represents relatively tractable cases that will be considered in some detail. Since the α-delayed Yule process, $\alpha = \beta^2$ contains the essential stochastic structure for applications to the β-field equations, the focus is on the former.

Recall that from (6.9) in the case $\alpha = \frac{1}{2}$, $N^{(\frac{1}{2})}(\tau) = \#V^{(\frac{1}{2})}(\tau)$ is a (shifted) Poisson process

$$P(N^{(\frac{1}{2})}(\tau) = k) = \frac{\tau^{k-1}}{(k-1)!}e^{-\tau},\ k \ge 1, \tau = \xi^2 t.$$

Let

$$p_n^{(\alpha)}(\tau) = P(N^{(\alpha)}(\tau) = 2^n, h^{(\alpha)}(\tau) = n)$$

denote the probability that an $\alpha = \beta^2$-tree originating at ξ is "fully saturated" by time t, i.e., has exactly n full generations (and 2^n branches); the saturated tree height h is defined as the maximal generation of the tree in which all nodes are present. Observe that for $n \geq 1$ the two subtrees resulting from the first branching must also be saturated with $n-1$ full generations, and therefore we have

$$p_0^{(\alpha)}(\xi^2 t) = e^{-\xi^2 t}$$

$$p_n^{(\alpha)}(\xi^2 t) = \xi^2 \int_0^t e^{-\xi^2(t-s)} \left[p_{n-1}^{(\alpha)}(\alpha \xi^2 s) \right]^2 ds, \qquad n \geq 1. \tag{6.11}$$

The case $\alpha = 1/2$ is amenable to exact calculations that will be useful in the analysis of regularity of solutions of the $\beta-$field equation. The main result is

Proposition 6.6. *For $n \geq 1$, let*

$$Q_n = \prod_{j=1}^{n} \left(\frac{1}{1 - \frac{1}{2^j}} \right)^{\frac{1}{2^j}}. \tag{6.12}$$

Then

$$p_n^{(\frac{1}{2})}(\xi^2 t) = e^{-\xi^2 t} \left(\frac{\xi^2 t}{2^n} \right)^{2^n - 1} Q_n^{2^n} \tag{6.13}$$

Proof. The statement is clearly true for $n = 1$. Assuming it holds for $n = k$, we have from (6.11)

$$\begin{aligned}
p_{k+1}^{(1/2)}(\xi^2 t) &= \xi^2 \int_0^t e^{-\xi^2(t-s)} \left[p_k^{(1/2)}(\xi^2 s/2) \right]^2 ds \\
&= \xi^2 \int_0^t e^{-\xi^2(t-s)} \left(e^{-\xi^2 s/2} \left(\frac{\xi^2 s/2}{2^k} \right)^{2^k - 1} Q_k^{2^k} \right)^2 ds \\
&= e^{-\xi^2 t} Q_k^{2^{k+1}} \left(\frac{1}{2^{k+1}} \right)^{2^{k+1} - 2} \int_0^{\xi^2 t} u^{2^{k+1} - 2} du \\
&= e^{-\xi^2 t} \left(\frac{1}{2^{k+1}} \right)^{2^{k+1} - 2} (\xi^2 t)^{2^{k+1} - 1} \frac{1}{2^{k+1} - 1} Q_k^{2^{k+1}} \\
&= e^{-\xi^2 t} \left(\frac{\xi^2 t}{2^{k+1}} \right)^{2^{k+1} - 1} \frac{1}{1 - \frac{1}{2^{k+1}}} Q_k^{2^{k+1}}
\end{aligned}$$

which is the statement for $n = k + 1$. □

Remark 6.3. In the context of binary tree searching, the constant $Q = \prod_{j=1}^{\infty}(1 - 1/2^j)$ is introduced.

Since $1 < Q_n < 1/Q$, and Q_n is increasing in n, $Q^* = \lim_{n\to\infty} Q_n$ is well defined. Furthermore, according to WolframAlpha [53],

$$Q^* = \prod_{k=1}^{\infty} \left(\frac{1}{1 - \frac{1}{2^k}} \right)^{\frac{1}{2^k}} \approx 1.55354. \tag{6.14}$$

6.1 Existence/Uniqueness of Mild Solutions

In this section we will pose the following question for both complex Burgers and β-field Burgers equations.

Remark 6.4. *Within the rather large literature, e.g.* [2, 18, 23, 28–31, 47, 49]*, on uniqueness/non-uniqueness to certain parabolic semi-linear equations associated with Markov branching processes, the explosion time distribution (and its complement) are known to play a key role in demonstrating non-uniqueness for initial data 0 (or 1, respectively). While the quasi-linear Burgers equation, and Navier-Stokes equations naturally involve semi-Markov branching processes in defining their genealogy, the consequences of explosion, or its absence, are not obvious for general initial data. Even in the case of the mean-field models, where the genealogy is a Markov branching process, the issues for general initial data are diverse; see* [15].

Existence/Uniqueness in $\mathcal{H}_\infty$: Does (3.1) and (4.4) have unique global in time mild solution for any $v_0 \in \mathcal{H}_\infty$?

We will give the detailed proofs for the case of Burgers equation, following Convention C. The corresponding proofs for β-field equations generally proceed similarly, and we will provide indications whenever necessary.

Existence of the solution to (3.8) and (4.3) hinges on finiteness of the expected value in (3.16), while the uniqueness is the consequence of the non-explosion property of the LJS-cascades for (complex) Burgers equation.

Proposition 6.7. *The expected value in (3.16) for the stochastic solution X defined either on Burgers or β-field cascades for $\beta \in [0, 1)$ is finite, provided $v_0(\xi) \in \mathcal{H}_\infty$. Thus $\hat{v}(\xi, t)$ provided by (3.16) is a well-defined solution to (3.8) and (4.3) respectively.*

Proof. Suppose $\|v_0\|_{\mathcal{H}_\infty} = M$. Using (6.5) we can estimate.

$$\mathbb{E} M^{N(\tau)} \le M \sum_{k=0}^{\infty} \frac{(M\tau)^k}{k!} e^{-\frac{\tau}{(k+1)}} \le e^{M\tau} < \infty. \tag{6.15}$$

Then $|\mathbb{E}X(\xi, t)| \le \mathbb{E}|X(\xi, t)| \le \mathbb{E}M^{N_{\text{unif}}(\xi^2 t)} < \infty$. The case of β-field Burgers equation with $\beta^2 \le 1/2$ is treated similarly.

In the case $\beta^2 > 1/2$ and $M > 2\beta^2 - 1$, estimate (6.10) only gives local existence:

$$\mathbb{E} M^{N_\beta(\tau)} \le M e^{-\tau} \sum_{k=0}^{\infty} \left(\frac{M(1 - e^{-(2\beta^2-1)\tau})}{2\beta^2 - 1} \right)^{n-1} = \frac{(2\beta^2 - 1) M e^{-\tau}}{(2\beta^2 - 1) - M(1 - e^{-(2\beta^2-1)\tau})}. \tag{6.16}$$

Clearly, $\mathbb{E}M^{N_\beta(\tau)} < \infty$ for $\tau \in [0, \tau_0]$, provided

$$\tau_0 < \frac{1}{2\beta^2 - 1} \ln\left(\frac{M}{M - (2\beta^2 - 1)}\right).$$

Let

$$C_0 = C_0(\beta, \tau_0) = \frac{(2\beta^2 - 1)}{(2\beta^2 - 1) - M(1 - e^{-(2\beta^2-1)\tau})}.$$

Clearly, since $M > 2\beta^2 - 1$ we have $C_0 > 1$.

Also note that $w(\tau) = \mathbb{E}M^{N_\beta(\tau)}$ solves the self-similar form of the β-field Burgers equation (6.24) with $w_0 = M$ on any interval containing zero where the expectation is finite.

The key observation is that we can use induction to extend $w(\tau)$ to a solution of (6.24), defined on the entire $[0, \infty)$. In fact any extension of the solution $w(t)$ extension will satisfy:

$$w(t) \le \gamma_n M e^{-\tau} = \left(\frac{2MC_0}{2\beta^2 - 1}\right)^{2^n} \frac{2\beta^2 - 1}{2} e^{-\tau}, \quad \text{for all } \tau \in \left[\frac{\tau_0}{(\beta^2)^{n-1}}, \frac{\tau_0}{(\beta^2)^n}\right], \ n \in \mathbb{N}, \tag{6.17}$$

where γ_n is defined by the following recursion:

$$\gamma_0 = C_0, \qquad \gamma_{n+1} = \frac{2M}{2\beta^2 - 1}\gamma_n^2, \qquad \text{i.e,} \qquad \gamma_n = \left(\frac{2M}{2\beta^2 - 1}\right)^{2^n - 1} C_0^{2^n}.$$

Indeed, in the case $n = 1$, we already have $w(t) \le C_0 M e^{-\tau}$ for $\tau \in [0, \tau_0]$. Now, assuming (6.17) holds for $k \le n$, we have for $\sigma \in [0, \tau_0/(\beta^2)^{n+1}]$, $\beta^2\sigma \in [0, \tau_0/(\beta^2)^n]$, and since γ_n is increasing, $w(\alpha\sigma) \le \gamma_n M e^{-\beta^2\sigma}$. Thus, for $\tau \in [0, \tau_0/(\beta^2)^{n+1}]$:

$$\begin{aligned} w(\tau) &= Me^{-\tau} + e^{-\tau}\int_0^\tau e^\sigma w^2(\beta^2\sigma)\, d\sigma \le Me^{-\tau}\left(1 + M\gamma_n^2 \int_0^\tau e^{-(2\beta^2-1)\sigma}\, d\sigma\right) \\ &\le \left(1 + \frac{M\gamma_n^2}{2\beta^2 - 1}\right) Me^{-\tau} \le \frac{2M}{2\beta^2 - 1}\gamma_n^2 M e^{-\tau} = \gamma_{n+1} M e^{-\tau}\ . \end{aligned}$$

Thus, (6.17) holds for all n.

As a consequence $\mathbb{E}M^{N_\beta(\tau)}$ does not blow up in finite "time" τ, and so for $\beta^2 > 1/2$, as in the other cases, $\mathbb{E}X_\beta(\xi, t)$ is a well-defined solution to (4.3).

□

Remark 6.5. The proof of Proposition 6.7 yields the following bounds on the growth rates for $\mathbb{E}M^{N_\bullet(\tau)}$, $M > 1$ as $\tau \to \infty$ (see Theorem 6.5), and consequently on the solutions of the corresponding equations with $\|v_0\|_{\mathcal{H}_\infty} = M$:

$$\mathbb{E}M^{N_\bullet(\tau)} \le O\left(Me^{c_\bullet M\tau}\right), \qquad \text{for complex Burgers or } \beta\text{-field Burgers, } \beta^2 \le (0, 1/2),$$

and

$$\mathbb{E}M^{N_\bullet(\tau)} \le O\left((c_\bullet M)^{\tau^{\frac{\ln 2}{\ln(1/\beta^2)}}}\right), \qquad \text{for } \beta\text{-field Burgers, } \beta^2 \in (1/2, 1),$$

where $c_\bullet$ is a constant that depends on the model, complex or β-field Burgers.

The uniqueness follows from the "martingale method" of Le Jan and Sznitman. Here it also requires the added non-explosion property established in Proposition 6.2. The proof is presented for sake of completeness.

Proposition 6.8. *Let $v(x,t)$ be the solution of (3.1) or (4.4) with $\beta \in [0,1)$ (equivalently, $\hat{v}(\xi,t)$ is a solution of (3.8) or (4.3)) with the initial data $v_0 \in \mathcal{H}_\infty$. Then $\hat{v}$ is is given by (3.16).*

Proof. Without loss of generality, suppose $\hat{v}(\xi,t)$ - a solution to (3.8), and X be the stochastic solution defined by (3.15). We aim to show $\hat{v} = \mathbb{E}X$. For this purpose we define recursively the following sequence: $X_0(\xi,t) = \hat{v}(\xi,t)$; Given $X_n(\xi,t)$, define X_{n+1} by

$$X_{n+1}(\xi,t) = \begin{cases} \hat{v}_0(\xi), & T_\theta \geq t \\ X_n(W_1(\xi), t - T_\theta) X_n(W_2(\xi), t - T_\theta), & T_\theta < t. \end{cases}$$

More explicitly,

$$X_n = \left(\prod_{\substack{|s| < n \\ s \in V_{\text{unif}}}} v_0(W_s) \right) \left(\prod_{\substack{|s| = n \\ \exists \tilde{s} \in V_{\text{unif}}\ s = \tilde{s}|n}} v(W_{s|n}, t - \sum_{j=0}^{n-1} T_{s|j}) \right).$$

By induction, it follows that

$$\mathbb{E}(X_n) = \hat{v}, \qquad \forall n \in \mathbb{N}.$$

Fix $\xi, t > 0$. Denote

$$M = \max\{ \max_{0 \leq \eta \leq \xi} \{|\hat{v}_0(\eta)|\}, \max_{\substack{0 \leq \eta \leq \xi, \\ 0 \leq s \leq t}} \{|\hat{v}(\eta, s)|\}\},$$

and let X_M be defined as in (3.15) but with $\hat{v}_0$ replaced with M. Note that since $M < \infty$, by Proposition 6.7, $\mathbb{E}(X_M) < \infty$.

Now let

$$A_n = \{s \in V_{\text{unif}} : |s| > n\}.$$

Then by the non-explosion,

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset,$$

and so by the dominated convergence theorem:

$$\mathbb{E}(\mathbf{1}_{A_n}) \to 0, \quad \text{as } n \to \infty.$$

Observe that $X_n|_{A_n^c} = X|_{A_n^c}$ and by the dominated convergence theorem, $\mathbb{E}(2X_M)\mathbf{1}_{A_n}) \to 0$ as $n \to \infty$. Therefore,

$$|\mathbb{E}(X_n) - \mathbb{E}(X)| \leq \mathbb{E}(2X_M)\mathbf{1}_{A_n}) \to 0 \quad \text{as } n \to \infty.$$

Thus, $\mathbb{E}(X) = \hat{v}$.

The β-field equation may be treated analogously. □

We collect the results above in the following theorem.

Theorem 6.1. *Consider either Burgers equation or β-field Burgers equation with $\beta \in [0,1)$ together with corresponding LJS-cascades. Then, for $\xi > 0$, with probability one*

- *No Stochastic Explosion:* $\zeta = \infty$.
- *∞-Radius of Convergence:* $\sup\{r \geq 1 : \mathbb{E}\, r^{N_\bullet(\xi,t)} < \infty\} = \infty, \quad t \geq 0.$

Moreover, for any $v_0(\xi) \in \mathcal{H}_\infty$

$$\hat{v}(\xi,t) = \mathbb{E}X(\xi,t), \quad \xi > 0, t \geq 0,$$

is the unique solution to (3.8) or (4.3).

Proof. The non-explosion is the consequence of Proposition 6.2, while existence and uniqueness are established in Propositions 6.7 and 6.8. The infinite radius of convergence for $\mathbb{E}\, r^{N_{\text{unif}}(\xi,t)}$ follows from (6.15) with $M = r$. □

6.2 Global Well-Posedness in $\mathcal{H}_\infty$.

As far as behavior at infinity, we will ask the following well-posedness question.

Well-posedness in $\mathcal{H}_\infty$: Suppose $v_0(x) \in \mathcal{H}_\infty$. If v denotes the solution to (3.1) or (4.4), will $v(x,t) \in \mathcal{H}_\infty$ for all $t > 0$?

It turns to that the answer is negative.

Theorem 6.2. *For either Burgers or β-field Burgers with $\beta \in [0,1)$ equations we have:*

1. *(Lack of well-posedness in $\mathcal{H}_\infty$) For any $M > 1$ there exist initial data v_0 with $\|v_0\|_{\mathcal{H}_\infty} \geq M$, namely,*
$$\hat{v}_0(\xi) \geq M > 1, \qquad \forall \xi \geq 0,$$
such that the corresponding solution $v(\xi,t)$ of (3.8) satisfies
$$\lim_{\xi\to\infty} \hat{v}(\xi,t) = \infty \qquad \forall t > 0,$$
i.e., Burgers equation (3.1) and β-field Burgers equations (4.4) are not well-posed in $\mathcal{H}_\infty$ even locally in time.

2. *(Well-posedness in $\mathcal{H}_\infty$ for small initial data). If $\|v_0\|_{\mathcal{H}_\infty} \leq 1$, then for all $t > 0$, the solution $v(\xi,t) \in \mathcal{H}_\infty$, i.e. the corresponding equations are globally well-posed in the unit ball of $\mathcal{H}_\infty$.*

Proof. The theorem follows immediately from Proposition 6.5 once we observe that in the case $\hat{v}_0(\xi) \geq M > 1$, $|X(\xi,t)| \geq M^{N_\bullet(\xi^2 t)}$, while in the case $\|v_0\|_{\mathcal{H}_\infty} \leq 1$, $|X(\xi,t)| \leq 1$. □

Remark 6.6. Note that for the self-similar initial data $\hat{v}_0(\xi) = M > 1$ the solution $\hat{v}(\xi, t) = w(\xi^2 t)$ satisfies

$$\lim_{\xi\to\infty} \hat{v}(\xi, t) = \infty \qquad \lim_{t\to\infty} \hat{v}(\xi, t) = \infty\,.$$

Remark 6.7. For $\beta = 1$ the global existence and uniqueness holds if and only if $\|v_0\|_{\mathcal{H}_\infty} \leq 1$. When $\|v_0\|_{\mathcal{H}_\infty} > 1$ the solutions in general do not exist even locally in time. Therefore, the corresponding β-field model is automatically ill-posed. Indeed, in this case β-field Burgers becomes

$$\frac{d\hat{v}(\xi, t)}{dt} = -\xi^2 \hat{v}(\xi, t) + \xi^2 \hat{v}^2(\xi, t),$$

and so

$$\hat{v}(\xi, t) = \frac{\hat{v}_0(\xi)}{\hat{v}_0(\xi) - e^{-\xi^2 t}(\hat{v}_0(\xi) - 1)}.$$

Clearly, if, e.g. $\hat{v}_0(\xi) \geq c > 1$, then for any $t > 0$, $\hat{v}(\xi, t)$ becomes infinite at a certain $\xi \in [0, \infty)$.

As will be shown in the next subsection, there is evidence that the lack of well-posedness in Theorem 6.2 for big initial data cannot be eliminated even if one considers smaller subspaces of $\mathcal{H}_\infty$. In fact, in the case $\beta = 1/\sqrt{2}$ there exist mild solutions of (4.4) with *compactly-supported* (in Fourier space) initial data that exit $\mathcal{H}_\infty$ in finite time.

6.3 Regularity

The analysis of regularity properties of the solution of the β-field model (4.3) can also be approached using the probabilistic representation of solutions given by

$$\hat{v}(\xi, t) = \mathbb{E}\left[\prod_{s\in V_\beta(\xi,t)} \hat{v}_0(\beta^{|s|}\xi)\right] \tag{6.18}$$

As already mentioned, for results pertaining to regularity of solutions, one may wish to consider bounded initial data having compact support on the positive half-line. For this let us consider

$$\hat{v}_0(\xi) = M\mathbf{1}_{[a,b]}, \quad 0 \leqslant a < b < \infty. \tag{6.19}$$

The particular case of $\beta = 1/\sqrt{2}$ serves to illustrate the lack of regularity in the solution of the corresponding β-field equation. Note that for the particular case that $\hat{v}_0(\xi) = M\mathbf{1}_{[\mathbf{0},\infty)}$ the solution is given by

$$\hat{v}(\xi, t) = \exp((M - 1)\, \xi^2 t)$$

This simple example shows that if $M < 1$, the solution gains regularity, indicated by an exponential decay in the Fourier domain but, that for $M > 1$, the solution leaves $\mathcal{H}_\infty$ instantly. This lack of well-posedness in the Hadamard sense is reminiscent of the behavior of solutions of the backward heat equation that is manifested also even in the case of initial data that is of compact support.

The precise statement of this result is as follows.

Proposition 6.9. *Let $\hat{v}(\xi,t)$ be a solution of (4.3) with $\beta = 1/\sqrt{2}$ and initial data $\hat{v}_0(\xi) = M\mathbf{1}_{[a,b]}$. Let $T_l = 1/b^2, T_u = 1/a^2$, and Q^* be given in (6.14). Then, if $M > e/Q^*$,*

$$\limsup_{\xi\to\infty} \hat{v}(\xi,t) = \infty, \text{ for } T_l < t < T_u.$$

Proof. : The probabilistic representation of the solution of the $\beta-$field model provides a lower bound that is fundamental to establishing the result. Fix $t \in [T_l, T_u]$ and let $\bar{\xi} = 1/\sqrt{t}$ so that by hypothesis, $\bar{\xi} \in [a,b]$.

Let $\xi_n = 2^{n/2}\bar{\xi}$. Then

$$\hat{v}(\xi_n,t) \geqslant p_n(2^n\bar{\xi}^2 t)M^{2^n}$$

where p_n is given by (6.13). Recalling the definition of Q_n given in (6.12) one has

$$\limsup_{n\to\infty} \hat{v}(\xi_n,t) \geqslant \lim_{n\to\infty} \left(e^{-1}MQ_n\right)^{2^n} = \infty$$

since Q_n is an increasing sequence, and we are assuming that $MQ^*e^{-1} > 1$. □

We note that the vanishing of the Fourier transform in a neighborhood of the origin plays a distinct role in this problem. Indeed, while the previous result shows that the solution leaves the space $\mathcal{H}_\infty$ in finite time, it does so for a finite time. To be precise,

Proposition 6.10. *Let $\hat{v}(\xi,t)$ be a solution of (4.3) with initial data $\hat{v}_0(\xi) = M\mathbf{1}_{[a,b]}$. and $\beta \in [0, 1/\sqrt{2}]$. Then if $a > 0$, there exists $T^*_\beta > 0$ such that*

$$\limsup_{\xi\to\infty} \hat{v}(\xi,t) = 0, \; \forall t > T^*_\beta.$$

Proof. The result follows by estimating the solution of the mild equation (4.3) on intervals of the form $\xi \in [a/\beta^k, a/\beta^{k+1}), k \geqslant 0$ and noting that for fixed t, the vanishing of the initial data near the origin imposes a limit on the number of branches that need to be consider.

For the particular initial data under consideration, (4.3) can be written as

$$\hat{v}(\xi,t) = e^{-\xi^2 t}M\mathbf{1}_{b>\xi>a} + \xi^2\int_0^t e^{-\xi^2(t-s)}\hat{v}^2(\beta\xi,s)\,ds. \tag{6.20}$$

Note that if $\xi \in [0,a)$, the solution vanishes, and if $\xi \in [a, a/\beta)$, $\hat{v}(\xi,t) = Me^{-\xi^2 t}$.

We consider first the case $\beta < 1/\sqrt{2}$. By induction one can show that if $a/\beta^n \le \xi < a/\beta^{n+1}$ then

$$\hat{v}(\xi,t) \le M\left(\frac{M}{1-2\alpha^2}\right)^{\gamma_n} e^{-(2\beta^2)^n\xi^2 t}, \tag{6.21}$$

where $\gamma_n = 2^n - 1$. Clearly the inequality holds for $n = 0$. Assuming the inequality holds for

$n = k$ and considering $a/\beta^{k+1} \le \xi < a/\beta^{k+2}$, we obtain:

$$\begin{aligned}
\hat{v}(\xi,t) &\le Me^{-\xi^2 t}\left(1+\frac{M^{2\gamma_k+1}}{(1-2\beta^2)^{2\gamma_k}}\xi^2\int_0^t e^{\xi^2 s}\left(e^{-(2\beta^2)^k(\beta\xi)^2 s}\right)^2 ds\right)\\
&= Me^{-\xi^2 t}\left(1+\frac{M^{\gamma_{k+1}}}{(1-2\beta^2)^{2\gamma_k}}\xi^2\int_0^t e^{(1-(2\beta^2)^{k+1})\xi^2 s}ds\right)\\
&= Me^{-\xi^2 t}\left(1+\frac{M^{\gamma_{k+1}}}{(1-2\beta^2)^{2\gamma_k}(1-(2\beta^2)^{k+1})}\left(e^{(1-(2\beta^2)^{k+1})\xi^2 t}-1\right)\right)\\
&\le Me^{-\xi^2 t}\frac{M^{\gamma_{k+1}}}{(1-2\beta^2)^{2\gamma_k+1}}e^{(1-(2\beta^2)^{k+1})\xi^2 t} = M\left(\frac{M}{1-2\beta^2}\right)^{\gamma_{k+1}} e^{-(2\beta^2)^{k+1}\xi^2 t},
\end{aligned}$$

and so (6.21) holds for $n = k+1$.

To complete the proof for $\beta < 1/\sqrt{2}$, note that for $\xi \in [a/\beta^n, a/\beta^{n+1})$ one has

$$\begin{aligned}
\hat{v}(\xi,t) &\le M\left(\frac{M}{1-2\beta^2}\right)^{\gamma_n}\exp\left(-(2\beta^2)^n\frac{a^2}{\beta^{2n}}t\right) = M\left(\frac{M}{1-2\beta^2}\right)^{2^n-1}\exp\left(-2^n a^2 t\right)\\
&= (1-2\beta^2)\left(\frac{Me^{-a^2 t}}{1-2\beta^2}\right)^{2^n}.
\end{aligned}$$

With

$$T^*_\beta = \frac{1}{a^2}\ln\left(\frac{M}{1-2\beta^2}\right)$$

one has for $t > T^*_\beta$ that

$$Me^{-a^2 t} < 1-2\beta^2$$

and the result follows since

$$\lim_{\xi\to\infty}\hat{v}(\xi,t) \le \lim_{n\to\infty}(1-2\beta^2)\left(\frac{Me^{-a^2 t}}{1-2\beta^2}\right)^{2^n} = 0$$

The case of $\beta = 1/\sqrt{2}$ is similar with (6.21) now replaced for $(\sqrt{2})^n a \le \xi < (\sqrt{2})^{n+1}a$, by

$$\hat{v}(\xi,t) \leqslant Me^{-\xi^2 t}\left(1+\frac{M\xi^2 t}{2^{n-1}}\right)^{\gamma_n} \tag{6.22}$$

where $\gamma_n = 2^n - 1$ as before. The statement holds for $n = 0$. Assume it holds for $n = k$ and consider $a(\sqrt{2})^{k+1} \le \xi < a(\sqrt{2})^{k+2}$. Then

$$\begin{aligned}
\hat{v}(\xi,t) &\le M e^{-\xi^2 t}\left(1 + M\xi^2 \int_0^t e^{\xi^2 s}\left(e^{-(\xi/\sqrt{2})^2 s}\right)^2 \left(1+\frac{M(\xi/\sqrt{2})^2 s}{2^{k-1}}\right)^{2\gamma_k} ds\right)\\
&= M e^{-\xi^2 t}\left(1 + M\xi^2 \int_0^t \left(1+\frac{M\xi^2 s}{2^k}\right)^{2\gamma_k} ds\right)\\
&= M e^{-\xi^2 t}\left(1 + \frac{2^k}{2\gamma_k+1}\left(1+\frac{M\xi^2 t}{2^k}\right)^{2\gamma_k+1} - \frac{2^k}{2\gamma_k+1}\right)\\
&\le M e^{-\xi^2 t}\left(1+\frac{M\xi^2 t}{2^k}\right)^{\gamma_{k+1}}.
\end{aligned}$$

and thus the inequality holds for $n = k+1$.

To complete the proof in this case, note that for $a(\sqrt{2})^n \le \xi < a(\sqrt{2})^{n+1}$ one has from (6.22)

$$\hat{v}(\xi,t) \le M e^{-2^n a^2 t}\left(1+4Ma^2 t\right)^{2^n-1} \le \frac{M}{1+4Ma^2 t}\exp\left[-2^n\left(a^2 t - \ln(1+4Ma^2 t)\right)\right].$$

Let $a^2 T^*_{1/\sqrt{2}}$ be the positive solution of the equation $s - \ln(1+4Ms) = 0$. Then the result follows since for $t > T^*_{1/\sqrt{2}}$,

$$\lim_{\xi\to\infty} \hat{v}(\xi,t) \le \lim_{n\to\infty} \exp\left[-2^n\left(a^2 t - \ln(1+4Ma^2 t)\right)\right] = 0$$

□

6.4 Self-Similarity

We note that the existence/uniqueness and well-posedness in $\mathcal{H}_\infty$ analysis of the LJS-cascades, in both complex Burgers and β-field Burgers equations exploited the natural scaling $(\xi,t) \to (\xi/\lambda, \lambda^2 t)$ in the crucial ways, most notably through the scaling-invariance of $V_\bullet$ and $N_\bullet$ in Propositions 6.2, 6.7, and 6.5. Clearly, self-similar solutions is this settings are the unique solutions that arise from the self-similar initial data $\hat{v}_0(\xi) = \hat{v}_0(\xi/\lambda)$, i.e. from *constant* $\hat{v}_0$. Thus, self-similar solutions present the limit-case scenarios for establishing existence (through the finiteness of the expected values (3.16)), uniqueness (through the non-explosion of the LJS-cascades) as well as lack of well-posedness (the most obvious ill-posed solutions are bounded below by self-similar solutions).

Remark 6.8. In the case $\hat{v}(\xi,t)$ is a self-similar solution, using the change of variables $\tau = \xi^2 t$ and setting $w(\tau) = v(\xi,t) = v(1,\tau)$ we obtain a *self-similar form* of the complex Burgers equation (3.8):

$$w(\tau) = w_0 e^{-\tau} + \int_0^\tau e^{-\sigma}\int_0^1 w(|\eta|^2(\tau-\sigma))w(|1-\eta|^2(\tau-\sigma))\,d\eta d\sigma, \qquad (6.23)$$

as well as a *self-similar from* of the β-field Burgers equations (4.3):

$$w(\tau) = w_0 e^{-\tau} + \int_0^\tau e^{-\sigma} w^2(\beta^2(\tau - \sigma))\, d\sigma. \tag{6.24}$$

Note that for $\alpha = \beta^2$, the last equation is a mild formulation of the following non-local differential equation:

$$u'(t) = -u(t) + u^2(\alpha t). \tag{6.25}$$

In [15] we refer to this equation as the *α-Riccati equation* and analyse it's LJS-cascades in the case $\alpha > 1$.

This close connection between well-posedness of self-similar and general (non-symmetric) solutions is more pronounced here than in the Navier-Stokes case treated in [13]. Viewed from the prism of symmetry breaking question:

Symmetry Breaking: Does the existence and uniqueness, or even well-posedness, of self-similar solutions differ from that of general *non self-similar* solutions in appropriate settings ?

For the Navier-Stokes case,[5] lack of symmetry breaking appeared on the level of LJS-cascades, which had the same finiteness and explosion properties for both self-similar and general formulations. For the Burgers equation in $\mathcal{H}_\infty$, Theorem 6.1 establishes that general solutions exhibit exactly the same properties as self-similar ones in this regard, and so there is *no symmetry breaking* in Burgers (nor in β-field Burgers) equations. The following is a stronger and more intriguing formulation of the question:

Symmetry and Regularity: Does well-posedness (or lack of it) of self-similar solutions in a natural scaling-invariant space mirror the persistence of regularity (or loss of it) for general solutions?

As we have seen in the case of β-field Burgers equation for $\beta = 1/\sqrt{2}$, the lack of well-posedness in $\mathcal{H}_\infty$ of self-similar solutions is correlated with a finite-time regularity loss for solutions arising from the smoothest possible initial data, albeit compactly supported in Fourier space. Thus it appears that, at least in this case, existence/uniqueness, well-posedness, *and* regularity properties of mild solutions are mirrored by the existence/uniqueness and well-posedness properties of the self-similar solutions.

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[5] The explosion problem for the self-similar LJS-cascade has been resolved in [16], where it has been shown that indeed explosion occurs.

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