A Division Algebra Description of the Magic Square, including E_8

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Division Algebras

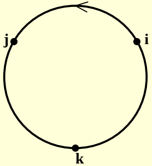
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

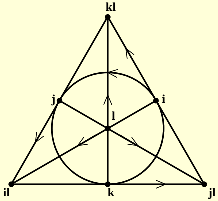
$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$

Split Octonions

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$$



$$\boxed{I^2 = J^2 = -U,\ L^2 = +U}$$

Split Division Algebras

$$\boxed{I^2 = J^2 = -U,\ L^2 = +U}$$

Signature $(4,4)$:

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

Overview

- $\mathfrak{e}_{8(-24)} = \mathfrak{su}(3, \mathbb{O}' \times \mathbb{O})$
  3×3 matrices
- $3 \times 3 \longmapsto 2 \times 2 + 2 \times 1$
  GUT $+$ spinors
- GUT: $\mathfrak{so}(12, 4) \supset \mathfrak{so}(3, 1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \otimes \mathbb{C}$
  Standard Model $+$ Lorentz
- Albert algebras $\subset \mathfrak{e}_8$

Next time: Standard Model

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Guiding Principle #1

Lie algebras are real!

(signature matters)

$\mathfrak{so}(3,1)$ has boosts and rotations

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

[Barton & Sudbery (2003), Wangberg (PhD 2007),
Dray & Manogue (CMUC 2010), Wangberg & Dray (JMP 2013, JAA 2014),
Dray, Manogue, & Wilson (CMUC 2014), Wilson, Dray, & Manogue (2022)]

The 2 × 2 Magic Square

Barton & Sudbery (2003):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014),
Dray, Huerta, & Kincaid (LMP 2014)]

Orthogonal Lie Algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

$$d = 3, 4, 6, 10$$

(1980s: Corrigan, Evans, Fairlie, Manogue, Sudbery)
(1990s: Manogue & Schray)

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

group: $P \longmapsto MPM^\dagger$ algebra: $P \longmapsto AP + PA^\dagger$

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (hermitian!): (so $P \longmapsto \{A, P\}$)

$$A = \sigma_x, \sigma_y, \sigma_z$$

$\mathfrak{so}(3,1)$

Vector in $\mathbb{C}' \oplus \mathbb{C}$

$$P = \begin{pmatrix} Lt + Uz & 1x - iy \\ 1x + iy & Lt - Uz \end{pmatrix}$$
$$= Lt\,\sigma_t + 1x\,\sigma_x + iy\,(-i\sigma_y) + Uz\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (antihermitian!): (so $P \longmapsto [A, P]$)

$$X_L = L\sigma_x, \quad X_{iL} = L\sigma_y, \quad D_L = L\sigma_z$$

$$\boxed{\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C}' \otimes \mathbb{C})}$$

From Clifford to Lorentz

Flips:

$\mathbf{Q} \longmapsto \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}$ reflects $\mathbf{Q}$ about $\mathbf{P}$.

Double Flips:

Successive flips about $\mathbf{P}_1$, $\mathbf{P}_2$ result in a (finite) rotation in the plane spanned by $\mathbf{P}_i$.

$$\boxed{\text{The } \textit{quadratic} \text{ elements of } \mathrm{C}\ell(p,q) \text{ generate } \mathrm{SO}(p,q)}$$

Nesting

Flips: $\mathbf{P} \longmapsto e_p \mathbf{P} e_p^{-1}$

Nested flips: $\mathbf{P} \longmapsto \mathbf{M}_2 \left(\mathbf{M}_1 \mathbf{P} \mathbf{M}_1^{-1}\right) \mathbf{M}_2^{-1}$

where

$$\begin{aligned}
\mathbf{M}_1 &= -e_p\, \mathbf{I} \\
\mathbf{M}_2 &= \left(e_p\, c(\tfrac{\theta}{2}) + e_q\, s(\tfrac{\theta}{2})\right)\, \mathbf{I} \\
&= \begin{cases} \left(e_p \cosh(\frac{\theta}{2}) + e_q\, \sinh(\frac{\theta}{2})\right) \mathbf{I}, & (e_p e_q)^2 = 1 \\ \left(e_p \cos(\frac{\theta}{2}) + e_q\, \sin(\frac{\theta}{2})\right) \mathbf{I}, & (e_p e_q)^2 = -1 \end{cases}
\end{aligned}$$

Theorem

The nested flips generate $SU(2, \mathbb{K}' \otimes \mathbb{K}) \cong SO(k + \frac{1}{2}k', \frac{1}{2}k')$

Summary: 2 × 2 Magic Square

- The algebras in the 2×2 magic square are $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 2×2 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & p \\ -\overline{p} & 0 \end{pmatrix}$$

Idea: rotations/boosts acting on $\mathbb{K}' \oplus \mathbb{K}$:

$$D_i = D_{1i};\ D_L = D_{UL};\ X_i = X_{iU};\ X_L = X_{1L}$$

- The remaining basis elements are of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 \\ 0 & i \circ j \end{pmatrix} = \frac{1}{2}\left[D_i, D_j\right]$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$.

Summary: 3×3 Magic Square

- The algebras in the 3×3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 3×3 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \cancel{S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}}, \quad X_p = \begin{pmatrix} 0 & p & 0 \\ -\overline{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\overline{p} & 0 \end{pmatrix}, \quad Z_p = \begin{pmatrix} 0 & 0 & -\overline{p} \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}$$

- The remaining basis elements ~~are~~ can be chosen to be of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 & 0 \\ 0 & i \circ j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$. **TRIALITY!**

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{\dagger-1} \quad \Longrightarrow \quad P \longmapsto MPM^\dagger,\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: Vector and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6$, $\mathcal{P} \in H_3(\mathbb{O})$

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ -\theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{\dagger-1} \quad \Longrightarrow \quad P \longmapsto MPM^\dagger,\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: ~~Vector~~ **Adjoint** and spinor actions at same time!

Example: $\mathcal{M} \in E_6,\ \mathcal{A} \in \mathfrak{e}_6, \mathcal{P} \in \mathfrak{e}_6$

Commutators

$$2 + 1 \Longrightarrow \mathfrak{e}_8 = \text{adjoint} + \text{spinors}$$

Adjoint action (commutators of rotations/boosts):

$$\mathfrak{so}(12,4) \longleftrightarrow X_q, D_p, D_{p,q}$$

$$\begin{aligned} D_i =& D_{1i}; \quad D_L = D_{UL}; \quad D_{i,j} = D_{i,j} \\ X_i =& X_{iU}; \quad X_L = X_{1L} \end{aligned}$$

Example: $[D_i, X_1] = [D_{1i}, X_{1U}] = 2X_{iU} = 2X_i$

Spinor action (possibly nested matrix multiplication):

$$\text{spinors} \longleftrightarrow Y_p, Z_q$$

$$Y_p + Z_q \longleftrightarrow \begin{pmatrix} -\overline{q} \\ p \end{pmatrix}$$

Example: $[D_i, Y_j] = -Y_k$

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!
- $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \mathbf{128}$.
- The $\mathbf{128}$ is a Majorana–Weyl representation of $\mathfrak{so}(12,4)$.
- The $\mathbf{128}$ contains spinor reps of each 2×2 algebra.

Guiding Principle #3

All representations live in $\mathfrak{e}_8$!

$$
\begin{aligned}
\mathfrak{e}_{8(-24)} &= \mathfrak{so}(12,4) \oplus \text{spinors} \\
\mathfrak{so}(12,4) &\supset \mathfrak{so}(3,1) \oplus \mathfrak{so}(7,3) \oplus \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(3,3) \oplus \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{su}(3)_c \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2)
\end{aligned}
$$

- $\mathfrak{so}(2)$ acts as complex structure in enveloping algebra (on spinors);
- $\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)$ is really $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{so}(1,1)$ ...
- ... but acts on spinors as $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ using complex structure.

Albert Algebra I

Albert algebra: 3×3 *Hermitian* matrices $\mathcal{A}$ over $\mathbb{O}$.
The Albert algebra is the minimal representation of $\mathfrak{e}_6$.

$$\mathfrak{e}_{8(-24)} = \mathfrak{e}_{6(-26)} \oplus 6 \times \mathbf{27} \oplus \mathfrak{sl}(3, \mathbb{R})$$

- The 6 of $\mathfrak{sl}(3, \mathbb{R})$ are "color labels": $\{I \pm IL, J \pm JL, K \pm KL\}$.
- Each **27** of $\mathfrak{e}_6$ *must be* an Albert algebra!
- $(K \pm KL)\mathcal{A}$ is *anti-Hermitian* over $\mathbb{O}' \otimes \mathbb{O}$ – and hence in $\mathfrak{e}_8$!
- Over $\mathbb{O}$, $(K \pm KL)\mathcal{I}$ is nested; really $\sim G_{K \pm KL} \in \mathfrak{g}_2'$.

[Dray, Manogue, Wilson (2023): A New Division Algebra Representation of E_6]

Two Subalgebras of $\mathbb{O}'$

$$\{I \pm IL, J \pm JL, K \mp KL\} \subset \mathbb{O}'$$

- These are 3-dimensional *subalgebras*!
- The only nonzero product is $(I \pm IL)(J \pm JL) = 2(K \mp KL)$.

Albert Algebra II

Jordan product:

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X})$$

Freudenthal product:

$$\begin{aligned}\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} &- \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})\,\mathcal{Y} + (\mathrm{tr}\mathcal{Y})\,\mathcal{X}\Big) \\ &+ \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})(\mathrm{tr}\mathcal{Y}) - \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I}\end{aligned}$$

Determinant:

$$\det(\mathcal{X}) = \frac{1}{3}\,\mathrm{tr}\Big((\mathcal{X} * \mathcal{X}) \circ \mathcal{X}\Big)$$

Idea: $\mathrm{tr}(\mathcal{X} \circ \mathcal{Y}) \longleftrightarrow \mathcal{X} \cdot \mathcal{Y}, \quad \mathcal{X} * \mathcal{Y} \longleftrightarrow \mathcal{X} \times \mathcal{Y}$

Albert Algebra III

"Dot":

$$[(K \pm KL)\mathcal{X}, (I \mp IL)\mathcal{Y}] = \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\, A_{J \pm JL}$$

"Cross":

$$[(I \pm IL)\mathcal{X}, (J \pm JL)\mathcal{Y}] = 4\,(K \mp KL)\,\mathcal{X} * \mathcal{Y}$$

[Dray, Manogue, Wilson (2023): A New Division Algebra Representation of E_7]

Albert Algebra and $\mathfrak{e}_7$

- $\mathfrak{e}_8 = \mathfrak{e}_7 \oplus 2 \times \mathbf{56} \oplus \mathfrak{su}(2)$
- $\mathfrak{e}_7$ is the conformalization of $\mathfrak{e}_6$, generated by $\mathfrak{e}_6$, two Albert algebras, and a dilation.
- Each **56** is a minimal representation of $\mathfrak{e}_7$, generated by two Albert algebras and two scalars.
- The action of $\mathfrak{e}_7$ on **56** uses the Freudenthal product and the trace of the Jordan product.

$\Longrightarrow$ These products *must* be realized as commutators in $\mathfrak{e}_8$!!

SUMMARY

Lie algebras are real!
The 3×3 structure is broken to 2×2.
All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \text{spinors}$$

$$\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \text{“} \otimes \mathbb{C}\text{”}$$

Albert algebras $\subset \mathfrak{e}_8$

- Wilson, Dray, and Manogue: An octonionic construction of E_8 ..., Innov. Incidence Geom. (in press), arXiv.org:2204.04996
- Dray, Manogue, and Wilson: A New ... Representation of E_6, arXiv.org:2309.?????
- Dray, Manogue, and Wilson: A New ... Representation of E_7, (in preparation)