A Division Algebra Description of the Magic Square, including E_8

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Division Algebras

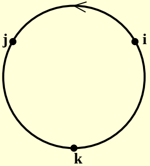
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

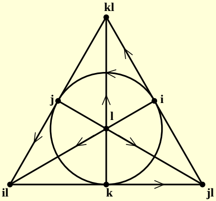
$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$

Split Octonions

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$$



$$I^2 = J^2 = -U, \; L^2 = +U$$

Split Division Algebras

$$I^2 = J^2 = -U,\ L^2 = +U$$

Signature $(4,4)$:

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$sa(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Description as symmetry groups

[Barton & Sudbery (2003), Wangberg (PhD 2007),
Dray & Manogue (CMUC 2010), Wangberg & Dray (JMP 2013, JAA 2014),
Dray, Manogue, & Wilson (CMUC 2014), Wilson, Dray, & Manogue (2022)]

Guiding Principle #1

Lie algebras are real!

(signature matters)

$\mathfrak{so}(3,1)$ has boosts and rotations

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

The 2 × 2 Magic Square

Barton & Sudbery (2003):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

"Vinberg":

$$sa(2, \mathbb{A} \otimes \mathbb{B}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{A}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{B})$$

$$\mathfrak{so}(\mathrm{Im}\,\mathbb{H}) = \mathfrak{so}(3); \quad \mathfrak{so}(\mathrm{Im}\,\mathbb{O}) = \mathfrak{so}(7)$$

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014), Dray, Huerta, & Kincaid (LMP 2014)]

Orthogonal ~~Groups~~ Lie Algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

$d = 3, 4, 6, 10$

(1980s: Corrigan, Evans, Fairlie, Manogue, Sudbery)

(1990s: Manogue & Schray)

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

group: $P \longmapsto MPM^\dagger$ algebra: $P \longmapsto AP + PA^\dagger$

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (hermitian!): (so $P \longmapsto \{A, P\}$)

$$A = \sigma_x, \sigma_y, \sigma_z$$

$\mathfrak{so}(3,1)$

Vector in $\mathbb{C}' \oplus \mathbb{C}$

$$P = \begin{pmatrix} Lt + Uz & 1x - iy \\ 1x + iy & Lt - Uz \end{pmatrix}$$
$$= Lt\,\sigma_t + 1x\,\sigma_x + iy\,(-i\sigma_y) + Uz\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (antihermitian!): (so $P \longmapsto [A, P]$)

$$X_L = L\sigma_x, \quad X_{iL} = L\sigma_y, \quad D_L = L\sigma_z$$

$$\boxed{\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C}' \otimes \mathbb{C})}$$

Summary: 2 × 2 Magic Square

- The algebras in the 2 × 2 magic square are $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 2 × 2 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & p \\ -\overline{p} & 0 \end{pmatrix}$$

Idea: rotations/boosts acting on $\mathbb{K}' \oplus \mathbb{K}$:

$$D_i = D_{1i};\ D_L = D_{UL};\ X_i = X_{iU};\ X_L = X_{1L}$$

- The remaining basis elements are of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 \\ 0 & i \circ j \end{pmatrix} = \frac{1}{2}\,[D_i, D_j]$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$.

The 3 × 3 Magic Square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{sp}(6,\mathbb{C})$	$\mathfrak{sp}(6,\mathbb{H})$	$\mathfrak{sp}(6,\mathbb{O})$
$\mathbb{O}'$	??	??	??	??

Dray & Manogue (2010):
$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_{6(-26)} \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{Spin}(9,1)$

Dray, Manogue, & Wilson (2014): $E_7 \cong \mathrm{Sp}(6,\mathbb{O})$

Minimal representation of $\mathfrak{e}_8$ is adjoint!

Guiding Principle #2

The 3 × 3 structure is broken to 2 × 2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{MPM}^{\dagger -1} \quad \Longrightarrow \quad P \longmapsto MPM^\dagger, \; \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P], \; \theta \longmapsto A\theta$$

Idea: Vector and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6$, $\mathcal{P} \in H_3(\mathbb{O})$

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ -\theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{\dagger -1} \quad \Longrightarrow \quad P \longmapsto MPM^\dagger,\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: ~~Vector~~ **Adjoint** and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6, \mathcal{P} \in \mathfrak{e}_6$

Summary: 3 × 3 Magic Square

- The algebras in the 3 × 3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 3 × 3 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \cancel{S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}}, \quad X_p = \begin{pmatrix} 0 & p & 0 \\ -\overline{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\overline{p} & 0 \end{pmatrix}, \quad Z_p = \begin{pmatrix} 0 & 0 & -\overline{p} \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}$$

- The remaining basis elements ~~are~~ can be chosen to be of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 & 0 \\ 0 & i \circ j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$. **TRIALITY!**

Commutators

$$2+1 \Longrightarrow \mathfrak{e}_8 = \text{adjoint} + \text{spinors}$$

Adjoint action (commutators of rotations/boosts):

$$\mathfrak{so}(12,4) \longleftrightarrow X_q, D_p, D_{p,q}$$

$$D_i = D_{1i}; \quad D_L = D_{UL}; \quad D_{i,j} = D_{i,j}$$
$$X_i = X_{iU}; \quad X_L = X_{1L}$$

Example: $[D_i, X_1] = [D_{1i}, X_{1U}] = 2X_{iU} = 2X_i$

Spinor action (possibly nested matrix multiplication):

$$\text{spinors} \longleftrightarrow Y_p, Z_q$$

$$Y_p + Z_q \longleftrightarrow \begin{pmatrix} -\overline{q} \\ p \end{pmatrix}$$

Example: $[D_i, Y_j] = -Y_k$

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!
- $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \mathbf{128}$.
- The **128** is a Majorana–Weyl representation of $\mathfrak{so}(12,4)$.
- The **128** contains spinor reps of each 2×2 algebra.

Guiding Principle #3

All representations live in $\mathfrak{e}_8$!

$$\begin{aligned}\mathfrak{e}_{8(-24)} &= \mathfrak{so}(12,4) + \text{spinors} \\ \mathfrak{so}(12,4) &\supset \mathfrak{so}(3,1) + \mathfrak{so}(7,3) + \mathfrak{so}(2) \\ &\supset \mathfrak{so}(3,1) + \mathfrak{so}(4) + \mathfrak{so}(3,3) + \mathfrak{so}(2) \\ &\supset \mathfrak{so}(3,1) + \mathfrak{su}(2)_L + \mathfrak{su}(2)_R + \mathfrak{su}(3)_c + \mathfrak{u}(1) + \mathfrak{so}(2)\end{aligned}$$

SUMMARY

Lie algebras are real!
The 3×3 structure is broken to 2×2.
All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \text{spinors}$$

$$\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) + \mathfrak{su}(3) + \mathfrak{su}(2) + \mathfrak{u}(1)$$

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