Using Octonions
to describe
the Standard Model

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(joint work with Robert Wilson)

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References

This work: `arXiv:2204.04996 & 2204.05310`

Our group:
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Others:
Jordan (1933), Jordan, von Neumann, & Wigner (1934), Freudenthal (1954, 1964), Tits (1966), Vinberg (1966), Gürsey, Ramond, & Sikivie (1976), Olive & West (1983), Kugo & Townsend (1983), Günaydin & Gürsey (1987), Chung & Sudbery (1987), Goddard, Nahm, Olive & Ruegg (1987), Corrigan & Hollowood (1988), Dixon (1994), Okubo (1995), Günaydin, Koepsell, & Nicolai (2001), Barton & Sudbery (2003), Cederwall (2007), Lisi (2007, 2010), Baez & Huerta (2010), Chester, Marran, & Rios (2021), Furey (2015), Furey & Hughes (2022ab)

Division Algebras

Real Numbers

$\mathbb{R}$

Division Algebras

Real Numbers

$$\mathbb{R}$$

Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

$$\boxed{i^2 = \qquad -1}$$

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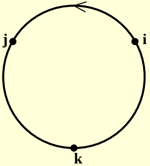
$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



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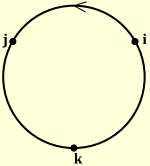

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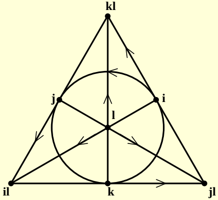
Octonions

$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$



$$i^2 = j^2 = \ell^2 = -1$$

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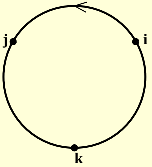

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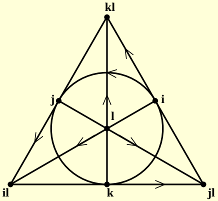
Split Octonions

$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$



$$I^2 = J^2 = -U, \; L^2 = +U$$

Split Division Algebras

$$I^2 = J^2 = -U,\; L^2 = +U$$

Signature $(4,4)$:

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

Lie Groups & Lie Algebras

Lie Group:

$$\mathrm{SO}(3) = \left\{ R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_x, R_y \right\}$$

Lie Algebra:

$$\mathfrak{so}(3) = \left\langle r_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_x, r_y \right\rangle$$

Properties:

$$R^\dagger = R^{-1}, \quad r_z = \left.\frac{dR_z}{d\theta}\right|_{\theta=0}, \quad r_z^\dagger = -r_z \quad [r_x, r_y] = r_z$$

Classification

Theorem (Cartan–Killing)

The only (simple) Lie algebras are (real forms of) $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$, *together with 5 exceptional cases:* $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

These are all unitary algebras!

$\mathfrak{so}(n) \cong \mathfrak{su}(n, \mathbb{R})$
$\mathfrak{su}(n) \cong \mathfrak{su}(n, \mathbb{C})$
$\mathfrak{sp}(n) \cong \mathfrak{su}(n, \mathbb{H})$
The exceptional cases are matrix algebras involving $\mathbb{O}$

The Tits–Freudenthal Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):
$F_4 \cong SU(3,\mathbb{O})$, $E_{6(-26)} \cong SL(3,\mathbb{O})$ using $SL(2,\mathbb{O}) \cong Spin(9,1)$

Dray, Manogue, & Wilson (2014): $E_7 \cong Sp(6,\mathbb{O})$

Wilson, Dray, & Manogue (2023): $E_8 \cong SU(3,\mathbb{O}'\otimes\mathbb{O})$

The algebras in the 3×3 magic square are $\mathfrak{su}(3,\mathbb{K}'\otimes\mathbb{K})$.

Spinors!

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ \theta^\dagger & n \end{pmatrix} \in \mathfrak{e}_8 \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in E_8$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{-1} \quad \Longrightarrow \quad P \longmapsto MPM^{-1}, \;\; \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P], \;\; \theta \longmapsto A\theta$$

$$(\mathcal{A} = \dot{\mathcal{M}}; \quad A = \dot{M})$$

Idea: Adjoint and spinor actions at same time!

2×2 Magic Square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

$$d = 3, 4, 6, 10$$

(1980s: Corrigan, Evans, Fairlie, Manogue, Sudbery)
(1990s: Manogue & Schray)

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014),
Dray, Huerta, & Kincaid (LMP 2014)]

Signature matters!

Lorentz Lie algebra: $\mathfrak{so}(3,1)$ [$\det P = -(-t^2 + x^2 + y^2 + z^2)$]

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

group: $P \longmapsto MPM^\dagger$ algebra: $P \longmapsto AP + PA^\dagger$

Signature matters!

Lorentz Lie algebra: $\mathfrak{so}(3,1)$ $\quad [\det P = -(-t^2 + x^2 + y^2 + z^2)]$

Vector in $\mathbb{C}' \oplus \mathbb{C}$

$$P = \begin{pmatrix} Lt + Uz & 1x - iy \\ 1x + iy & Lt - Uz \end{pmatrix}$$

$$= Lt\,\sigma_t + 1x\,\sigma_x + iy\,(-i\sigma_y) + Uz\,\sigma_z$$

Rotations (antihermitian!): $\quad$ (so $P \longmapsto [A, P]$)

$$X_i = i\sigma_x, \quad X_1 = i\sigma_y, \quad D_i = i\sigma_z$$

Boosts (antihermitian!): $\quad$ (so $P \longmapsto [A, P]$)

$$X_L = L\sigma_x, \quad X_{iL} = L\sigma_y, \quad D_L = L\sigma_z$$

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!

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$$\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) \oplus \ldots$$

The Standard Model

Fermions		**Bosons**	
Leptons (Dirac spinors)		Mediators (Vectors)	
e^-, μ^-, τ^-	charge $= -1$	γ	$\mathfrak{u}(1)$
ν_e, ν_μ, ν_τ	charge $= 0$	$W^\pm, Z$	$\mathfrak{su}(2)$
Quarks (Dirac spinors)			
u, c, t	charge $= \frac{2}{3}$	gluons	$\mathfrak{su}(3)$
d, s, b	charge $= -\frac{1}{3}$		
		Higgs (scalar)	

Generations:

3 copies that differ only by mass

Dirac Spinors

- Solutions of the Dirac equation
- Represent leptons and quarks
- *Two* Weyl spinors of opposite chirality
  $(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \cong \mathfrak{so}(4))$
- $\mathfrak{su}(2)_L$ acts only on one chirality for all fermions

GUTs

Is there a (semi-)simple group that contains $\mathbf{U}(1) \times \mathbf{SU}(2)_L \times \mathbf{SU}(3)$?

Common candidates are SU(5) and SO(10).

Lie algebras are real!
The 3×3 structure is broken to 2×2.
All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \text{spinors}$$

$$\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \text{“} \otimes \mathbb{C} \text{”}$$

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