E_8 and the Standard Model

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References

This work: `arXiv:2204.04996 & 2204.05310`

Our group:
Fairlie & Manogue (1986, 1987), Manogue & Sudbery (1989), Schray (PhD 1994), Manogue & Schray (1993), Dray & Manogue (1998ab, 1999), Manogue & Dray (1999), Dray, Janesky, & Manogue (2000), Dray, Manogue, & Okubo (2002), Dray & Manogue (CAA 2000, CMUC 2010), Manogue & Dray (2010), Wangberg (PhD 2007), Wangberg & Dray (JMP 2013, JAA 2014), Dray, Manogue, & Wilson (CMUC 2014), Kincaid (MS 2012), Kincaid and Dray (MPLA 2014), Dray, Huerta, & Kincaid (LMP 2014)

Others:
Jordan (1933), Jordan, von Neumann, & Wigner (1934), Freudenthal (1954, 1964), Tits (1966), Vinberg (1966), Gürsey, Ramond, & Sikivie (1976), Olive & West (1983), Kugo & Townsend (1983), Günaydin & Gürsey (1987), Chung & Sudbery (1987), Goddard, Nahm, Olive & Ruegg (1987), Corrigan & Hollowood (1988), Dixon (1994), Okubo (1995), Günaydin, Koepsell, & Nicolai (2001), Barton & Sudbery (2003), Cederwall (2007), Lisi (2007, 2010), Baez & Huerta (2010), Chester, Marran, & Rios (2021), Furey (2015), Furey & Hughes (2022ab)

Research Agenda

- Standard Model in $\mathfrak{e}_{8(-24)}$ or ~~$\mathfrak{e}_{8(8)}$~~ or ~~$\mathfrak{e}_8$~~.
  (symmetries, fermions, mediators, ~~gravity~~, ~~super-strings/symmetry~~)
- Use mathematician's conventions. (anti-Hermitian!)
- Pay attention to signature. ($\mathfrak{e}_8$ is **real**)

Magic Squares

2 × 2 "half-split": (spin groups ...)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

3 × 3 "half-split": (... plus spinor reps)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

3 × 3 Structure

$$\left(\begin{array}{c|c} \mathfrak{so}(12,4) & \text{spinor} \\ \hline -\text{spinor}^\dagger & 0 \end{array}\right)$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Algebra	Maximal Subalgebra	Centralizer
$\mathfrak{so}(3)$	$\mathfrak{so}(2)$	$\mathfrak{g}_2 \oplus \mathfrak{g}_{2(2)}$
$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{so}(2,1) \oplus \mathfrak{so}(1,1)$	$\mathfrak{g}_2 \oplus \mathfrak{sl}(3,\mathbb{R})$
$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{so}(5,1) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{sl}(3,\mathbb{R})$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1)$	$\mathfrak{sl}(3,\mathbb{R})$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(8,4) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2)$
$\mathfrak{e}_{8(-24)}$	$\mathfrak{so}(12,4)$	—

Research Agenda Addition

- Standard Model in $\mathfrak{e}_{8(-24)}$ or ~~$\mathfrak{e}_{8(8)}$~~ or ~~$\mathfrak{e}_8$~~.
  (symmetries, fermions, mediators, ~~gravity~~, ~~super-strings/symmetry~~)
- Use mathematician's conventions. (anti-Hermitian!)
- Pay attention to signature. ($\mathfrak{e}_8$ is **real**)
- Decompose into subalgebras and centralizers.

Decomposition of $\mathfrak{so}(12,4)$

$$\mathfrak{e}_{8(-24)} \supset \mathfrak{so}(12,4) + \overbrace{\text{spinors}}^{\text{complex}}$$

$$\mathfrak{so}(12,4) \supset \underbrace{\mathfrak{so}(2)}_{\substack{\text{complex}\\ \text{structure}}} + \mathfrak{so}(3,1) + \underbrace{\mathfrak{so}(7,3)}_{\text{“}\mathfrak{so}(10)\text{”}}$$

Lorentz + $\mathfrak{so}(10)$ GUT

Spinors are complex

Decompositions of $\mathfrak{so}(n)$ vs. $\mathfrak{su}(n)$

$$\mathfrak{so}(p+q) = \mathfrak{so}(p) + \mathfrak{so}(q) + p \times q$$
$$\mathfrak{su}(p+q) = \mathfrak{su}(p) + \mathfrak{su}(q) + 2 \times p \times q + \mathfrak{u}(1)$$

$$\mathfrak{so}(p+1) = \mathfrak{so}(p) + p$$
$$\mathfrak{su}(p+1) = \mathfrak{su}(p) + 2 \times p + \mathfrak{u}(1)$$

$$\mathfrak{so}(6) \cong \mathfrak{su}(4) = \mathfrak{su}(3) + 2 \times 3 + \mathfrak{u}(1)$$

Color Decomposition of $\mathfrak{e}_8$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

$$\mathfrak{e}_{8(-24)} = \begin{cases} \cancel{\mathfrak{e}_{6(2)} + 54 \times 3 + \overbrace{\mathfrak{su}(3)}^{\text{compact}}} \\ \mathfrak{e}_{6(-26)} + \underbrace{27}_{\text{Jordan algebra}} \times 6 + \mathfrak{sl}(3,\mathbb{R}) \end{cases}$$

Everyone (including us) who studied Jordan algebra representations of $\mathfrak{e}_6$ was almost correct...

Decomposition of $\mathfrak{e}_8 \longrightarrow \mathfrak{e}_6$

$$\mathfrak{so}(12,4) = \mathfrak{so}(9,1) + 10 \times 6 + \overbrace{\mathfrak{so}(3,3)}^{\mathfrak{sl}(3,\mathbb{R}) + 3 \times 2 + \langle S_L \rangle}$$

Majorana–Weyl spinors of both $\mathfrak{so}(9,1)$ & $\mathfrak{so}(3,3)$

$$128 = 16 \times (1 + 1 + 3 + \overline{3})$$

$$\mathfrak{e}_{8(-24)} = \underbrace{\mathfrak{e}_{6(-26)}} + \underbrace{27 \times 6} + \mathfrak{sl}(3,\mathbb{R})$$

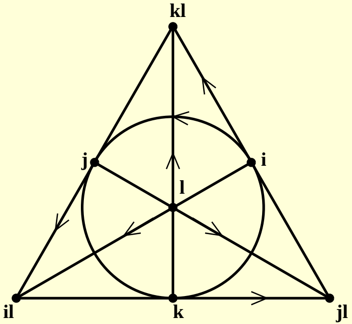
$\mathfrak{so}(9,1)$
$16 \times 2 \times 1$
S_L

$10 \times 2 \times 3$
$16 \times 2 \times 3$
$1 \times 2 \times 3$

Research Agenda Addition

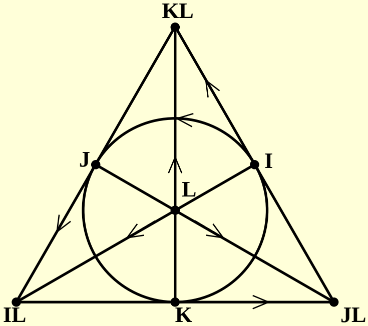
- Standard Model in $\mathfrak{e}_{8(-24)}$ or ~~$\mathfrak{e}_{8(8)}$~~ or ~~$\mathfrak{e}_8$~~.
  (symmetries, fermions, mediators, ~~gravity~~, ~~super-strings/symmetry~~)
- Use mathematician's conventions. (anti-Hermitian!)
- Pay attention to signature. ($\mathfrak{e}_8$ is **real**)
- Decompose into subalgebras and centralizers.
- Label everything with division algebras.

Octonions



1

$\mathbb{O}$



U

$\mathbb{O}'$

(reverse arrows if 2 Ls)

Basis

$$\left(\begin{array}{c|c} \mathfrak{so}(12,4) & \text{spinor} \\ \hline -\text{spinor}^\dagger & 0 \end{array}\right) \longleftrightarrow \left(\begin{array}{cc|c} D & X & -Z^\dagger \\ -X^\dagger & D & Y \\ \hline Z & -Y^\dagger & 0 \end{array}\right)$$

$\mathfrak{so}(12,4)$: Basis consists of rotations/boosts in single plane:
(boosts contain L, IL, JL, KL)

$D_{i,j}$, $D_L = D_{UL}$ (both indices in $\mathbb{O}$, or both in $\mathbb{O}'$)

$X_{iL}, X_j = X_{jU}$ (one index in each)

Spinors: (like X)

Y_{iL}, Z_{iL} (one index in each)

Commutator Rules

Adjoint $\mathfrak{so}(12,4)$:

$$[D_{i,j}, D_{j,k}] = -2D_{i,k} \qquad \ldots$$

Spinors:

$$[D_{i,j}, Y_{aA}] = Y_{i(ja)A} \qquad \ldots$$

Killing Duals

Normalized Killing form B; our basis is orthonormal.
(rotations/boosts have squared norm $+1/-1$)

$$S_L = D_{I,IL} + D_{J,JL} + D_{K,KL} = \begin{pmatrix} L & & \\ & L & \\ & & -2L \end{pmatrix}$$

$(3,3)$-component! Triality!
Spinors split into Killing pairs, e.g. $Y_{1\pm L}$.

Research Agenda Addition

- Standard Model in $\mathfrak{e}_{8(-24)}$ or ~~$\mathfrak{e}_{8(8)}$~~ or ~~$\mathfrak{e}_8$~~.
  (symmetries, fermions, mediators, ~~gravity~~, ~~super-strings/symmetry~~)
- Use mathematician's conventions. (anti-Hermitian!)
- Pay attention to signature. ($\mathfrak{e}_8$ is **real**)
- Decompose into subalgebras and centralizers.
- Label everything with division algebras.
- Put leptons and electroweak theory into $\mathfrak{e}_{6(-26)}$ ($\mathbb{C}' \otimes \mathbb{O}$)

  Labels: U, L; 1, i, j, k, $k\ell$, $j\ell$, $i\ell$, ℓ

Spinors of $\mathfrak{e}_6$: Volume Elements

$$\mathfrak{e}_{6(-26)} = \mathfrak{so}(9,1) + \overbrace{16 \times 2}^{\text{Killing duals}} + \langle S_L \rangle$$

$$\mathfrak{so}(9,1) \longmapsto \mathfrak{so}(2) + \mathfrak{so}(7,1) \longmapsto \mathfrak{so}(2) + \overbrace{\mathfrak{so}(3,1)}^{\text{Lorentz}} + \underbrace{\overbrace{\mathfrak{so}(4)}^{\text{weak}}}_{\mathfrak{su}(2)_L + \mathfrak{su}(2)_R}$$

Labels on Adjoint Elements

$\mathfrak{so}(3,1)$	$D_{i,j}$	$D_{j,k}$	$D_{k,i}$
(i,j,k,L)	X_{kL}	X_{iL}	X_{jL}
$\mathfrak{so}(4)$	$D_{i\ell,j\ell}$	$D_{j\ell,k\ell}$	$D_{k\ell,i\ell}$
$(i\ell,j\ell,k\ell,\ell)$	$D_{k\ell,\ell}$	$D_{j\ell,\ell}$	$D_{i\ell,\ell}$

$\mathfrak{su}(2)_L$	$\mathfrak{su}(2)_R$
$D_{i\ell,j\ell} - D_{k\ell,\ell}$	$D_{i\ell,j\ell} + D_{k\ell,\ell}$
$D_{j\ell,k\ell} - D_{i\ell,\ell}$	$D_{j\ell,k\ell} + D_{i\ell,\ell}$
$D_{k\ell,i\ell} - D_{j\ell,\ell}$	$D_{k\ell,i\ell} + D_{j\ell,\ell}$

Labels on Spinors

spin: $[D_{i,j}, [X_1, \psi]]$

weak: $[D_{i\ell,j\ell} + D_{k\ell,\ell}, [X_1, \psi]]$

Lorentz (Weyl) Spinor

weak/spin	
$+/+$	$Y_{1(1+L)} + Z_{k(1-L)}$ $-Y_{k(1+L)} - Z_{1(1-L)}$
$+/-$	$Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} + Z_{j(1-L)}$
$-/+$	$-Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} - Z_{j(1-L)}$
$-/-$	$Y_{1(1+L)} - Z_{k(1-L)}$ $Y_{k(1+L)} - Z_{1(1-L)}$

ℓ in label $\longleftrightarrow$ left-handed

$\mathfrak{su}(2)_L$ annihilates right-handed spinors

$\mathfrak{su}(2)_R$ annihilates left-handed spinors

Labels on Spinors

Quark

weak/spin	
$+/+$	$Y_{1(J-JL)} - Z_{k(J-JL)}$ $-Y_{k(J-JL)} + Z_{1(J-JL)}$
$+/-$	$Y_{j(J-JL)} - Z_{i(J-JL)}$ $-Y_{i(J-JL)} - Z_{j(J-JL)}$

Lepton

weak/spin	
$+/+$	$Y_{1(1+L)} + Z_{k(1-L)}$ $-Y_{k(1+L)} - Z_{1(1-L)}$
$+/-$	$Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} + Z_{j(1-L)}$

$L \longleftrightarrow -L$ interchanges Killing duals

$$\left[S_L, Z_{a(1\pm L)}\right] = \pm 3 Z_{a(1\pm L)}$$

$$\left[S_L, Z_{a(J\pm JL)}\right] = \mp Z_{a(J\pm JL)}$$

(charge as usual from $\mathfrak{so}(10)$ as sum of Cartan elements)

Labels on Spinors

spin: $\left[D_{i,j}, [X_1, \psi]\right]$

weak: $\left[D_{i\ell,j\ell} + D_{k\ell,\ell}, [X_1, \psi]\right]$

Lorentz (Weyl) Spinor

weak/spin	
$+/+$	$Y_{1(1+L)} + Z_{k(1-L)}$ $-Y_{k(1+L)} - Z_{1(1-L)}$
$+/-$	$Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} + Z_{j(1-L)}$
$-/+$	$-Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} - Z_{j(1-L)}$
$-/-$	$Y_{1(1+L)} - Z_{k(1-L)}$ $Y_{k(1+L)} - Z_{1(1-L)}$

ℓ in label $\longleftrightarrow$ left-handed

$\mathfrak{su}(2)_L$ annihilates right-handed spinors

$\mathfrak{su}(2)_R$ annihilates left-handed spinors

Generations

$\mathfrak{su}(2)_R$

$D_{i\ell,j\ell} + D_{k\ell,\ell}$
$D_{j\ell,k\ell} + D_{i\ell,\ell}$
$D_{k\ell,i\ell} + D_{j\ell,\ell}$

Choose different Cartan element!
Mixes up eigenstates:
$k \longmapsto i \longmapsto j \longmapsto k$
Exactly 3 generations!

Lorentz (Weyl) Spinor

weak/spin	
$+/+$	$Y_{1(1+L)} + Z_{k(1-L)}$ $-Y_{k(1+L)} - Z_{1(1-L)}$
$+/-$	$Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} + Z_{j(1-L)}$
$-/+$	$-Y_{j(1+L)} + Z_{i(1-L)}$ $-Y_{i(1+L)} - Z_{j(1-L)}$
$-/-$	$Y_{1(1+L)} - Z_{k(1-L)}$ $Y_{k(1+L)} - Z_{1(1-L)}$

Mediators

Lie algebra = Cartan subalgebra + raising/lowering operators

$$\mathfrak{so}(7,1) = \mathfrak{so}(3,1) + \mathfrak{so}(4) + 4 \times 4$$

4 × 4 has both Lorentz and weak vector labels.

$$\begin{matrix} X_{\ell L} & D_{\ell,i} & D_{\ell,j} & D_{\ell,k} \\ X_{k\ell L} & D_{k\ell,i} & D_{k\ell,j} & D_{k\ell,k} \\ X_{j\ell L} & D_{j\ell,i} & D_{j\ell,j} & D_{j\ell,k} \\ X_{i\ell L} & D_{i\ell,i} & D_{i\ell,j} & D_{i\ell,k} \end{matrix}$$

$$\mathfrak{so}(6,4) = \mathfrak{so}(3,1) + \mathfrak{so}(3,3) + 4 \times 6$$

4 × 6 has both Lorentz and color vector labels.

$\Longrightarrow$ 6 gluons in a vector representation

SUMMARY

Everything in $\mathfrak{e}_{8(-24)}$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \text{spinors}$$

$$= \underbrace{\mathfrak{e}_{6(-26)}}_{\mathfrak{so}(9,1) + 2\times 16 + \langle S_L\rangle} + 27\times 6 + \mathfrak{sl}(3,\mathbb{R})$$

gluons & quarks

$$\underbrace{\mathfrak{so}(9,1)}_{} + 2\times 16 + \langle S_L\rangle$$

leptons

$$\mathfrak{so}(2) + \mathfrak{so}(3,1) + \underbrace{\mathfrak{su}(2)_L + \mathfrak{su}(2)_R}_{\text{3 generations}} + \underbrace{4\times 4}_{\text{electroweak mediators}}$$

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