Labels

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\{1, U\}$	$\{\ldots, k\}$	$\{\ldots, i, j\}$	$\{\ldots, i\ell, j\ell, k\ell, \ell\}$
$\mathbb{C}'$	$\{\ldots, L\}$			
$\mathbb{H}'$	$\{\ldots, K, KL\}$			
$\mathbb{O}'$	$\{\ldots, I, IL, J, JL\}$			

New Description of $\mathfrak{e}_8$
(Wilson et al.)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

- Everything is 3x3 "matrices" with two "labels"
- Ordinary matrices/commutators in quaternionic cases.
- Generalize commutators for double-labeled diagonal elements.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

Orthogonal Lie Algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

+Spinors

- 2x2 Lie algebras are degree 2 in Clifford algebra.
- 2x2->3x3 adds spinor representations with appropriate Bott periodicity.

Type Structure

$$\begin{pmatrix} D & X & -Z^\dagger \\ -X^\dagger & \pm D & Y \\ Z & -Y^\dagger & 0 \end{pmatrix}$$

- *D*s must have both labels in the same division algebra. (We don't always write {1, *U*}).
- *X*s, *Y*s, *Z*s have one label in each division algebra.

Choices for Octions Models

- Everything in real $\mathfrak{e}_8$
  - The minimal representation is the adjoint, so actors and actees are in the same space.
  - Don’t complexify, pay attention to signature.
  - Always stay in the Magic Square.
- Prioritize Lorentz, weak, color over generation.
- No gravity.
- Allow Clifford and Jordan algebra structures to emerge from $\mathfrak{e}_8$.

Rules of Today's Game

- Pick an entry in the magic square.
- Assign division algebra labels.
- Decompose into smaller entry and centralizer.
- Interpret *D*s and *X*s as adjoints/bosonic reps.
- Interpret the *Y*s and *Z*s as fermionic reps.

Example: $\mathfrak{a}_{5(-7)}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\{1, U\}$	$\{\ldots, k\}$	$\{\ldots, i, j\}$	$\{\ldots, i\ell, j\ell, k\ell, \ell\}$
$\mathbb{C}'$	$\{\ldots, L\}$		$\mathfrak{a}_{5(-7)}$	
$\mathbb{H}'$	$\{\ldots, K, KL\}$			
$\mathbb{O}'$	$\{\ldots, I, IL, J, JL\}$			

Content of: $\mathfrak{a}_{5(-7)}$

- Adjoint $\mathfrak{so}(5,1)$ (labels $\{U, L, 1, i, j, k\}$)
- Adjoint $\mathfrak{su}(2)$

$$GS_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$$

- Adjoint $\mathfrak{so}(1,1)$

$$S_L = \begin{pmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & -2L \end{pmatrix}$$

- 2 spinor 8s of $\mathfrak{so}(5,1)$ (labels $\{U \pm L, 1, i, j, k\}$)

Extension: $\mathfrak{e}_{6(-26)}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\{1, U\}$	$\{\ldots, k\}$	$\{\ldots, i, j\}$	$\{\ldots, i\ell, j\ell, k\ell, \ell\}$
$\mathbb{C}'$	$\{\ldots, L\}$		$\mathfrak{a}_{5(-7)}$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\{\ldots, K, KL\}$			
$\mathbb{O}'$	$\{\ldots, I, IL, J, JL\}$			

Content of: $\mathfrak{e}_{6(-26)}$

- Add labels in $\mathbb{H}_\perp$ i.e. $\{i\ell, j\ell, k\ell, \ell\}$
- $\mathfrak{e}_6 = \mathfrak{a}_5 \oplus \mathfrak{su}(2) \oplus 40$
- Lorentz structures inside representations of $\mathfrak{a}_5$
  - 4 more $\mathfrak{so}(5,1)$ Lorentz vectors with labels in $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
  - Double the number of spinors (16) by including those with labels in $\mathbb{H}_\perp$

Rewrite Quaternionic Matrices

$$GS_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$D_{\ell,k\ell} - D_{i\ell,j\ell}$$

$$D_{\ell,i\ell} - D_{j\ell,k\ell}$$

$$D_{\ell,j\ell} - D_{k\ell,i\ell}$$

$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Old	New
$D_{\ell,k\ell} - D_{i\ell,j\ell}$	$D_{\ell,k\ell} + D_{i\ell,j\ell}$
$D_{\ell,i\ell} - D_{j\ell,k\ell}$	$D_{\ell,i\ell} + D_{j\ell,k\ell}$
$D_{\ell,j\ell} - D_{k\ell,i\ell}$	$D_{\ell,j\ell} + D_{k\ell,i\ell}$

Handedness

- $\mathfrak{e}_6$ is our first example with octonionic content.
- $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ all with $\mathbb{H}_\perp$ labels.
- New $\mathfrak{su}(2)$ annihilates the old spinors.
- Old $\mathfrak{su}(2)$ annihilates the new spinors.
- Second Weyl handedness of $\mathfrak{so}(5,1)$ spinors emerges from non-associativity of octonions

Decompositions

- $\mathfrak{so}(p+q) = \mathfrak{so}(p) \oplus \mathfrak{so}(q) \oplus p \times q$

- The spinors decompose appropriately.

Extension: $\mathfrak{e}_{8(-24)}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\{1, U\}$	$\{\ldots, k\}$	$\{\ldots, i, j\}$	$\{\ldots, i\ell, j\ell, k\ell, \ell\}$
$\mathbb{C}'$	$\{\ldots, L\}$			
$\mathbb{H}'$	$\{\ldots, K, KL\}$			
$\mathbb{O}'$	$\{\ldots, I, IL, J, JL\}$			$\mathfrak{e}_{8(-24)}$

Content of: $\mathfrak{e}_{8(-24)}$

- $\mathfrak{e}_8 = \mathfrak{e}_6 \oplus \mathfrak{sl}(3,\mathbb{R}) \oplus 27\times 3 \oplus \overline{27}\times\overline{3}$
- Add Labels $\{I \pm IL, J \pm JL, K \pm KL\}$
- 27 is literally a Jordan algebra times a null label => all the old work about $\mathfrak{e}_6$ acting on Jordan algebras applies straightforwardly.

Take Home Messages

If you break $\mathfrak{e}_8$ in these ways:

- $\mathfrak{a}_5$ gives chirality, right-handed leptons, and $\mathfrak{su}(2)$
- $\mathfrak{e}_6$ adds left-handed leptons, another $\mathfrak{su}(2)$ and potential weak mediators.
- $\mathfrak{e}_8$ adds colored $\left(3+\overline{3}\right)$ Jordan 27s to build quarks/baryons with potentially testable properties.
- $\mathfrak{e}_7$ shows how to build determinants of the 27 representations of $\mathfrak{e}_6$ using color.