The Octonionic Eigenvalue Problem

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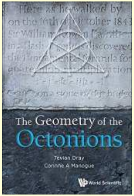
Book



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http://octonions.geometryof.org/GO

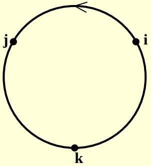
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



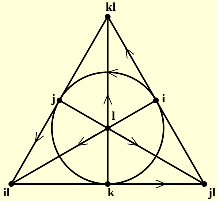
Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$



$$i^2 = j^2 = \ell^2 = -1$$

Cayley–Dickson (1919)

Noncommutative:

$$ji = -ij$$

Nonassociative:

$$(ij)\ell = -i(j\ell)$$

Norm:

$$|x|^2 = x\overline{x}$$

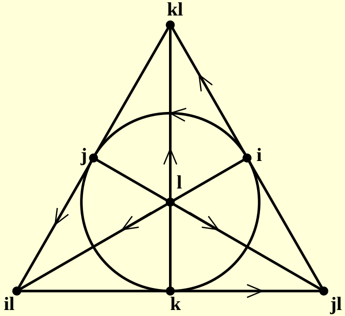
$$|x| = 0 \implies x = 0$$

Composition:

$$|xy| = |x||y|$$

Inverses (Division!):

$$x \neq 0 \implies x^{-1} = \overline{x}/|x|^2$$



The Standard Eigenvalue Problem

$$Av = \lambda v$$
$$(A^\dagger = A)$$

Reality: $\lambda \in \mathbb{R}$

Existence: $\exists n$ eigenvalues (counting multiplicity)

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

Orthonormal Basis: $\exists$ orthonormal basis of eigenvectors

Decomposition: $A = \sum \lambda_m v_m v_m^\dagger$

Reality: $(\mathbf{A}^\dagger = \mathbf{A} \implies \overline{\lambda} = \lambda)$

$$Av = \lambda v \quad \implies \quad \overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v \neq \lambda v^\dagger v$$

$$Av = v\lambda \quad \implies \quad \overline{\lambda}(v^\dagger v) \neq (Av)^\dagger v \neq v^\dagger (Av) \neq (v^\dagger v)\lambda$$

Orthogonality: $(\lambda_1 \neq \lambda_2 \implies \mathbf{v}_1^\dagger \mathbf{v}_2 = 0)$

$$Av_m = \lambda_m v_m \implies \lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 \neq v_1^\dagger (Av_2) = \lambda_2 v_1^\dagger v_2$$

Theorem (Dray & Manogue 1998)

$v \in \mathbb{O}^3, \mathcal{A}^\dagger = \mathcal{A} \in \mathbf{h}(3, \mathbb{O}), \lambda \in \mathbb{R} \implies$

- $\exists$ **6** $(= 2 \times 3)$ *real eigenvalues* λ_m, *with* $\mathcal{A} v_m = \lambda_m v_m$;
- $(v_m v_m^\dagger) v_n = 0$ *for* $m \neq n$ *in the same "family".*

Example $(\lambda \neq \mathbb{R})$

$$\mathbf{A} = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad v = \begin{pmatrix} j \\ k\ell \end{pmatrix} \implies \mathbf{A} v = vi$$

The Right Eigenvalue Problem

$$Av = v\lambda$$
$$(A^\dagger = A)$$

Reality: Over $\mathbb{H}$, $\lambda \in \mathbb{R}$, but not over $\mathbb{O}$

Existence: 3×3 matrices over $\mathbb{O}$ have 2×3 real eigenvalues

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow (v_1 v_1^\dagger) v_2 = 0$

Orthonormal Basis: $\exists$ 2 orthonormal bases of eigenvectors

Decomposition: $A = \sum \lambda_m (v_m v_m^\dagger)$ $(\times 2)$

Characteristic Equation

$$\mathcal{A} \in \mathbf{h}(3, \mathbb{O})$$

$\Longrightarrow$

$$\mathcal{A}^3 - (\mathrm{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det\mathcal{A})\,\mathcal{I} = 0$$

but

$$\lambda^3 - (\mathrm{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - (\det\mathcal{A}) = r_m$$

- Matrix solves characteristic equation;
- Eigenvalues do not;
- $\exists\, 2$ “families” of eigenvalues.

Characteristic Operator

$$\mathcal{A} = \begin{pmatrix} x & a & \overline{c} \\ \overline{a} & y & b \\ c & \overline{b} & z \end{pmatrix}$$

$(v \in \mathbb{O}^3,\ q \in \mathbb{O})$

$$K[v] = \mathcal{A}\big(\mathcal{A}(\mathcal{A}v)\big) - (\mathrm{tr}\mathcal{A})\,\mathcal{A}(\mathcal{A}v) + \sigma(\mathcal{A})\,\mathcal{A}v - (\det \mathcal{A})v$$

$\Longrightarrow K$ diagonal $\longmapsto$

$$K[q] = c(b(aq)) + \overline{a}(\overline{b}(\overline{c}q)) - \Big(c(ba) + (\overline{a}\overline{b})\overline{c}\Big)q$$

"Family" structure of $\mathbb{O}$

(Dray, Manogue, & Okubo 2002)

$$\mathbb{T} = \langle 1, a, b, c \rangle \subset \mathbb{O} \qquad \longleftrightarrow \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$$\Phi = \mathrm{Re}(a \times b \times c) = \frac{1}{2}\,\mathrm{Re}\big(a(\bar{b}c) - c(\bar{b}a)\big)$$

$$= \mathrm{Im}(a) \cdot [\mathrm{Im}(b) \times \mathrm{Im}(c)] \qquad \text{(triple product)}$$

$$\alpha = [a, b, c] = (ab)c - a(bc) \qquad \text{(associator)}$$

$$K[q] = r_m q \iff q \in \mathbb{T}_m \subset \mathbb{O}; \quad r_m^2 - 4\Phi r_m - \alpha^2 = 0$$

$$\mathbb{T}_m = \mathbb{T} s_m; \quad s_m = \frac{r_m + 4\Phi + \alpha}{2(r_m + 2\Phi)}$$

$$\mathbb{O} = \mathbb{T} s_1 \oplus \mathbb{T} s_2 \qquad (s_1 + s_2 = 1)$$

$$\mathbb{T}_2 \equiv \mathbb{T}_1 \alpha \qquad (\mathbb{T}_1 \perp \mathbb{T}_2)$$

$$x, y \in \mathbb{T}_m \Longrightarrow x\bar{y} \in \mathbb{T}$$

The Jordan Eigenvalue Problem

(Dray & Manogue 1999)

$$\mathcal{A} \in \mathbf{h}(3, \mathbb{O})$$

$$\mathcal{V} \circ \mathcal{V} = \mathcal{V}$$

$$\mathcal{A} \circ \mathcal{B} = (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})/2$$

$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V}$$

Equivalent to right eigenvalue problem over $\mathbb{H}$! $(\mathcal{V} = vv^\dagger)$

$$(vv^\dagger) \circ (vv^\dagger) = (v^\dagger v)(vv^\dagger)$$

- usual characteristic equation
- $\lambda \in \mathbb{R}$
- Cayley–Moufang plane ($\mathbb{OP}^2$)
- Solutions of 10-d Dirac equation!

Dirac equation

Position space: $(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu})$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\hat{\Psi} = 0$$

Momentum space: $(\Psi = e^{-ip_\nu x^\nu/\hbar}\hat{\Psi})$

$$(\gamma^\mu p_\mu - mc)\Psi = 0$$

Weyl equation ($m = 0$): [up to normalization]

$$\mathbf{P}v = 0 \Longrightarrow \mathbf{P} = \widetilde{vv^\dagger} \Longrightarrow \begin{pmatrix} \widetilde{\mathbf{P}} & v \\ v^\dagger & 1 \end{pmatrix} \in \mathbb{OP}^2$$

Works in 3,4,6,10 spacetime dimensions! Supersymmetry!!

Octonionic projections are quaternionic!

$(a, b, c \in \mathbb{O};\ x, y, z \in \mathbb{R})$

$$\mathcal{A} = \begin{pmatrix} x & a & \overline{c} \\ \overline{a} & y & b \\ c & \overline{b} & z \end{pmatrix}$$

$$\mathcal{A}^2 = \begin{pmatrix} x^2 + |a|^2 + |c|^2 & (x+y)a + \overline{c}\,\overline{b} & (x+z)\overline{c} + ab \\ (x+y)\overline{a} + bc & |a|^2 + y^2 + |b|^2 & (y+z)b + \overline{ac} \\ (x+z)c + \overline{b}\,\overline{a} & (y+z)\overline{b} + ca & |c|^2 + |b|^2 + z^2 \end{pmatrix}$$

$$\mathcal{A}^2 = \mathcal{A} \Longrightarrow ab = (1 - x - z)\,\overline{c} \Longrightarrow [a, b, c] = 0!$$

Application: Solutions to 10-d Dirac equation (octonionic) are in fact 6-d (quaternionic), leaving room for additional symmetry.

Simultaneous Eigenstates

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

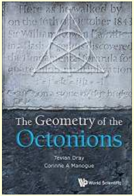
$$L_m\psi := -\frac{\hbar}{2}(\ell\sigma_m\psi)\ell$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow \begin{aligned} 2\,\mathbf{L_z}\psi &= \hbar\,\psi \\ 2\,\mathbf{L_x}\psi &= -\hbar\,\psi\,\mathbf{k} \\ 2\,\mathbf{L_y}\psi &= -\hbar\,\psi\,\mathbf{k}\boldsymbol{\ell} \end{aligned}$$

"spin-up" is simultaneous eigenstate of L_x, L_y, L_z!
(but **eigenvalues** don't commute!)

SUMMARY

- Real eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Always get decompositions into primitive idempotents;
- Splits octonions into two "almost quaternionic" subspaces!
- Jordan eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Primitive idempotents are quaternionic! ($\mathbb{OP}^2$)
- Applications to physics: spin, Dirac equation...



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The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$\mathit{sa}(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Description as symmetry groups

[Wangberg (PhD 2007), Wangberg & Dray (JMP 2013, JAA 2014), Dray, Manogue, and Wilson (CMUC 2014)]

History

- Barton & Sudbery (2003):
  Well-understood in terms of Lie algebras.
- Satisfactory group description not yet known.
- Rosenfeld (1956/1997):
  Isometry groups of projective planes over $\mathbb{A} \otimes \mathbb{B}$.

  Cayley-Moufang plane: $F_4 \longleftrightarrow \mathbb{OP}^2$

- Baez (2002):
  OK for E_6; not for E_7, E_8.

  In short, more work must be done before we can claim to fully understand the geometrical meaning of the Lie groups E_6, E_7 and E_8.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_6 \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{SO}(9,1) \subset E_6$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Triality!

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{sp}(6,\mathbb{C})$	$\mathfrak{sp}(6,\mathbb{H})$	$\mathfrak{sp}(6,\mathbb{O})$
$\mathbb{O}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_{6(-26)} \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{SO}(9,1)$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Triality!

Dray, Manogue, Wilson (2014):

$E_{7(-25)} \cong \mathrm{Sp}(6,\mathbb{O})$

The Subgroup Structure of E_6

116

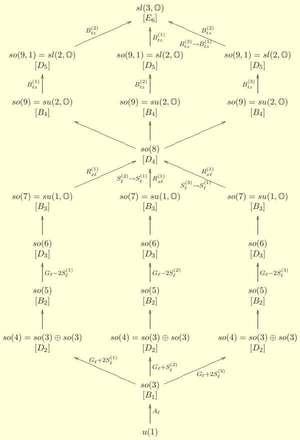


Figure 4.2: Chain of subgroups $SO(n) \subset SO(9,1) \subset SL(3,\mathbb{O})$

164

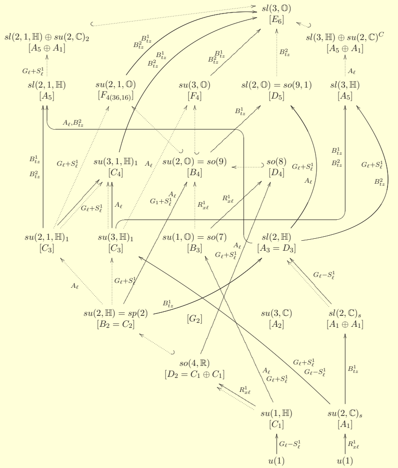


Figure 5.1: Preferred subalgebra chains of E_6 using the same basis

Wangberg (PhD 2007), Wangberg & Dray (JAA 2014)

Cartan Decompositions of E_6

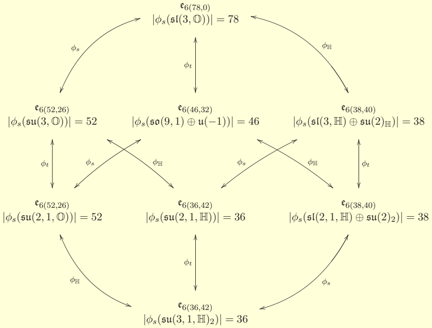


FIG. 3. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

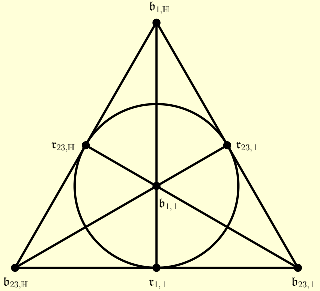


FIG. 4. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

Wangberg (PhD 2007), Wangberg & Dray (JMP 2013)

The 2 × 2 Magic Square

Barton & Sudbery (2003):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

"Vinberg":

$$sa(2, \mathbb{A} \otimes \mathbb{B}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{A}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{B})$$

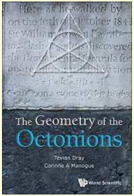
$$\mathfrak{so}(\mathrm{Im}\,\mathbb{H}) = \mathfrak{so}(3); \quad \mathfrak{so}(\mathrm{Im}\,\mathbb{O}) = \mathfrak{so}(7)$$

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014), Dray, Kincaid, & Huerta (LMP 2014)]

SUMMARY

- **Have:** $E_6 \cong \mathrm{SL}(3, \mathbb{O})$
  [Dray & Manogue (2010)]
- **Have:** Structure of E_6
  [Wangberg (PhD 2007), Wangberg & Dray (2013; 2014)]
- **Have:** 2×2 Magic Square
  [Kincaid (MS 2012), Kincaid and Dray (2014),
  Dray, Kincaid, & Huerta (2014)]
- **(Mostly) Have:** $E_7 \cong \mathrm{Sp}(6, \mathbb{O})$
  [Dray, Manogue, Wilson (CMUC 2014)]
- **Want:** $E_8 \cong ??$



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