The Geometry of Special Relativity

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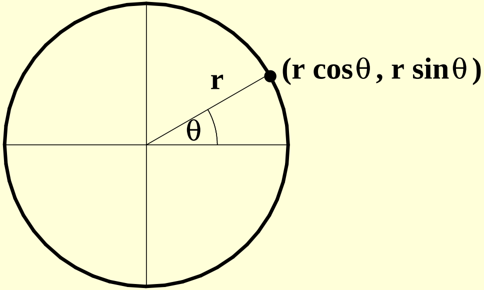
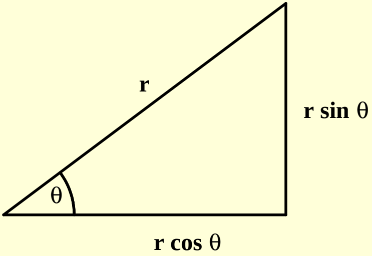
Harry S. Kieval Lecture

20 May 2025

Circle Geometry



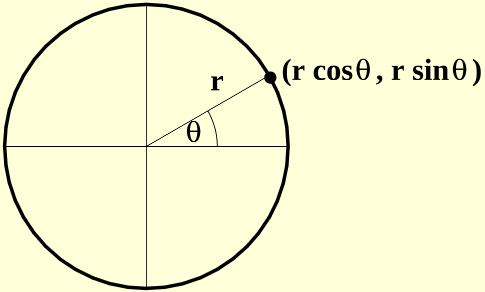


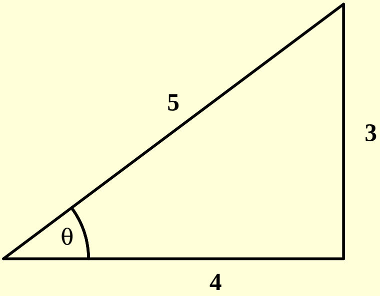
$x^2 + y^2 = r^2$

$r\theta =$ arclength

(θ in radians)

If $\tan\theta = \frac{3}{4}$, what is $\cos\theta$?

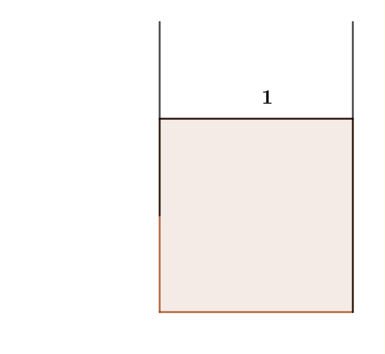


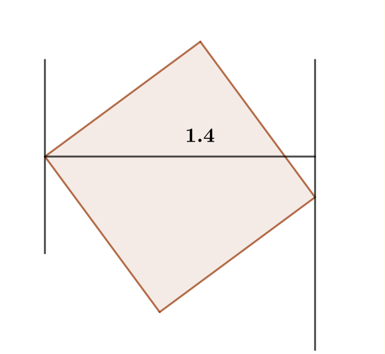


$$x^2 + y^2 = r^2$$
$$r\theta = \text{arclength}$$
$(\theta$ in radians$)$

$$\tan\theta = \frac{3}{4} \implies \cos\theta = \frac{4}{5}$$

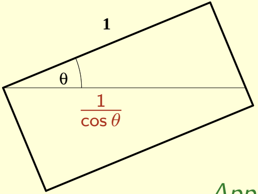
Measurements

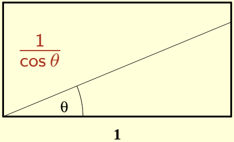




Measurements

Width:



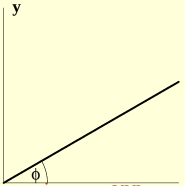


Apparent width > 1

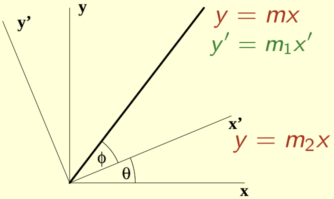
(since $\theta \in (0, \frac{\pi}{2}) \Longrightarrow \cos\theta < 1$*)*

Slope: $y = mx$

How do slopes add?

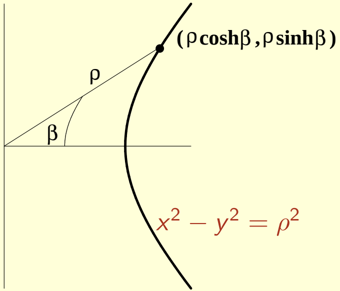


$$m_1 = \frac{\text{rise}}{\text{run}} = \frac{\text{opp}}{\text{adj}} = \tan(\phi)$$



$$m = \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

Hyperbola Geometry



$\rho\beta = \text{arclength}$

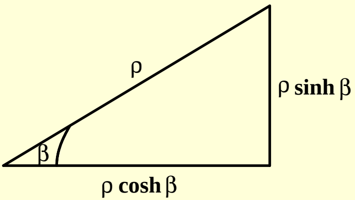
$ds^2 = |dx^2 - dy^2|$

$\cosh\beta = \frac{1}{2}\left(e^{\beta} + e^{-\beta}\right) \geq 1$

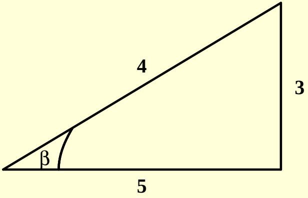
$\sinh\beta = \frac{1}{2}\left(e^{\beta} - e^{-\beta}\right)$

Hyperbolic Triangle Trig

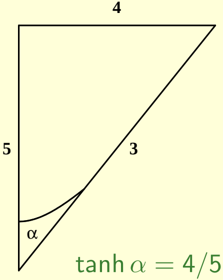
$\cosh^2\beta - \sinh^2\beta = 1$



Draw a 3–4–5 triangle in hyperbola geometry.

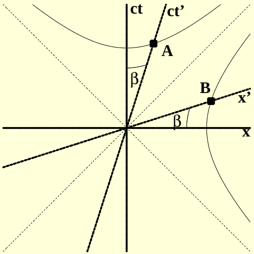


$\tanh\beta = 3/5$



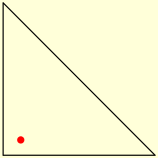
$\tanh\alpha = 4/5$

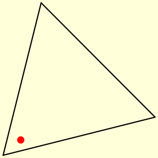
Special Relativity

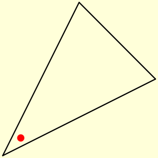


$$x^2 - ct^2 = x'^2 - ct'^2$$
$$(c = 1)$$

Draw a right triangle in hyperbola geometry.



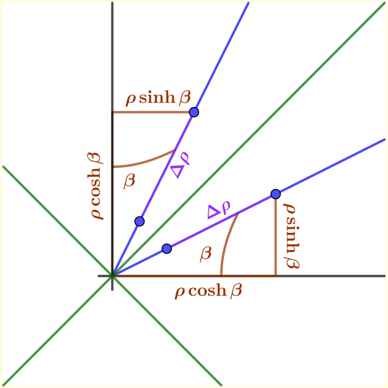




"right angles" are not angles!

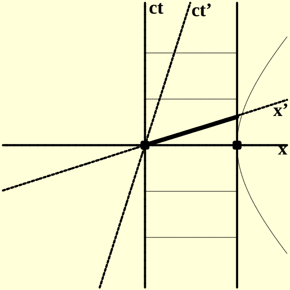
Drawing Spacetime Diagrams

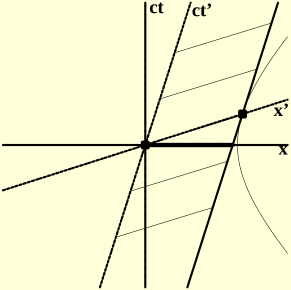
- Points in spacetime are called *events*.
- Slope $m = \pm 1$ represents beams of light.
- Vertical lines represent objects at rest.
- Horizontal lines represents simultaneous events (in the given reference frame).
- Slope $|m| > 1$ (*timelike*) represents observer moving at constant speed.
- Speed is given by $c \tanh \beta$, where β is (hyperbolic) angle from a *vertical* line.
- The "distance" between two events on such a line is the time between them measured by the moving observer.
- Slope $|m| < 1$ (*spacelike*) represents simultaneous events for observer moving at constant speed.
- The distance between two events on such a line is the distance between them measured by the moving observer.



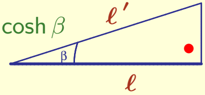
Length Contraction

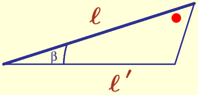
Draw a spacetime diagram showing a meter stick at rest.





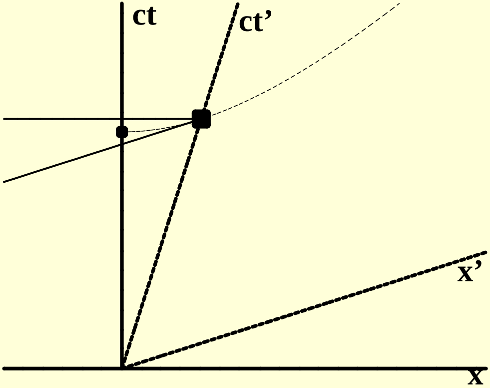
$\ell = \ell' \cosh\beta$





$$\ell' = \frac{\ell}{\cosh\beta}$$

Time Dilation



Cosmic Rays

The collision of cosmic rays with gas nuclei 60 km above the surface of the earth produces μ-mesions, whose half-life before decaying into other particles is 1.5 μs $= 1.5 \times 10^{-6}$s. Even at the speed of light, it would take

$$\frac{60\ \text{km}}{3 \times 10^8\ \frac{\text{m}}{\text{s}}} = 200\ \mu\text{s}$$

to reach the surface, which is

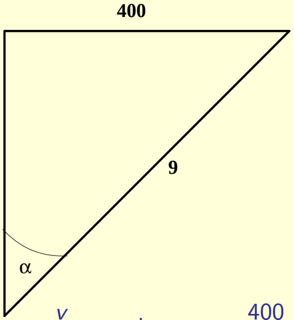
$$\frac{200\ \mu\text{s}}{\frac{3}{2}\ \mu\text{s per half-life}} = \frac{400}{3}\ \text{half-lives}$$

which is long enough that almost none would survive the journey. In actual fact, roughly $\frac{1}{8}$ of the mesons reach the earth!

(So only 3 half-lives.)

How fast are they going?

$$\frac{(60\ \text{km})(1000\ \frac{\text{m}}{\text{km}})}{3(1.5 \times 10^{-6}\ \text{s})(3 \times 10^8\ \frac{\text{m}}{\text{s}})} = \frac{400}{9}$$

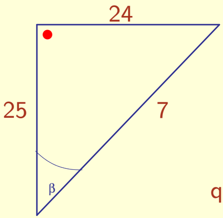


$$\frac{v}{c} = \tanh\alpha = \frac{400}{\sqrt{400^2 + 9^2}}$$

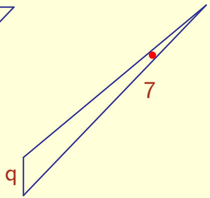
$$\approx .99974697$$

(based on Taylor & Wheeler, **1st edition**, Ex. 42, p. 89.)

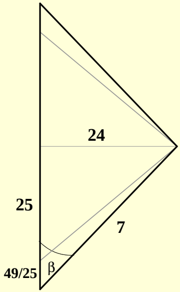
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$



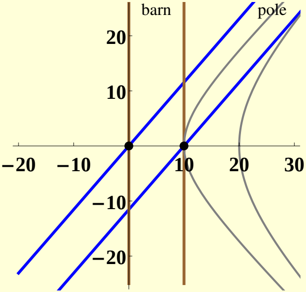
$$q = \frac{7}{\cosh\beta} = \frac{49}{25}$$



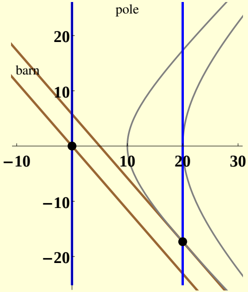
Straight path takes longest!

The Pole and the Barn

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn? Draw a spacetime diagram!



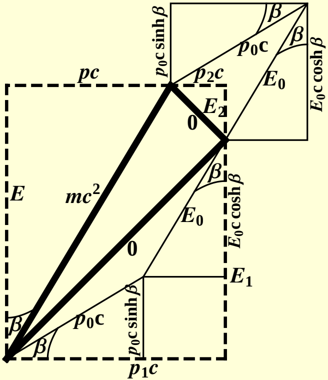
BARN

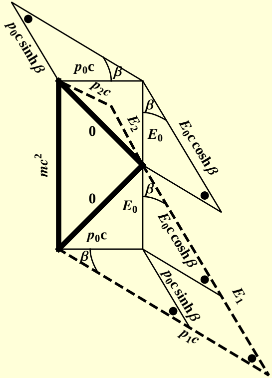


POLE

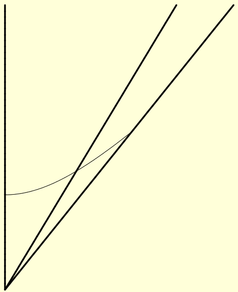
Relativistic Mechanics

A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon. $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$





Addition of Velocities



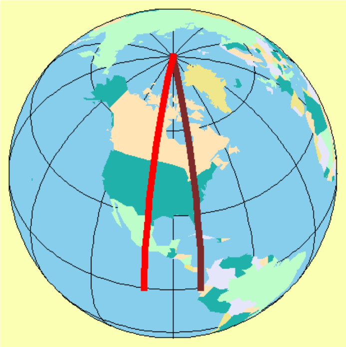
$$\frac{v}{c} = \tanh\beta$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

Einstein addition formula!

Which Geometry?

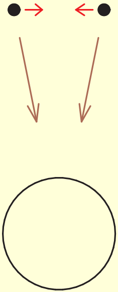
signature	flat	curved
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	



Tidal forces!

Which Geometry?

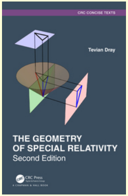
signature	**flat**	**curved**
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	Lorentzian



General Relativity!

(Need calculus to describe curvature!)

SUMMARY



https://math.oregonstate.edu/~tevian
https://relativity.geometryof.org/GSR

Special Relativity = Hyperbolic Trigonometry

General Relativity = Lorentzian Vector Calculus

THE END