THE GEOMETRY OF SPECIAL RELATIVITY

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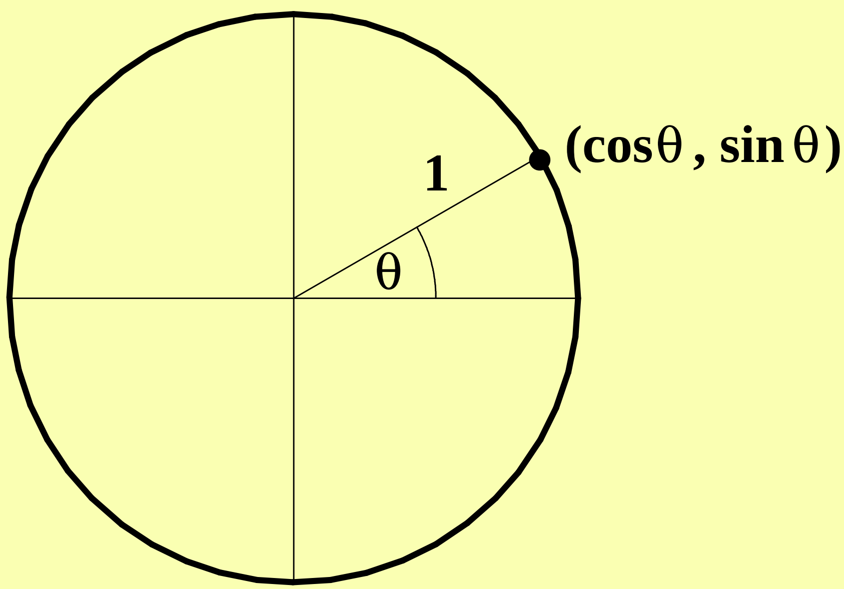
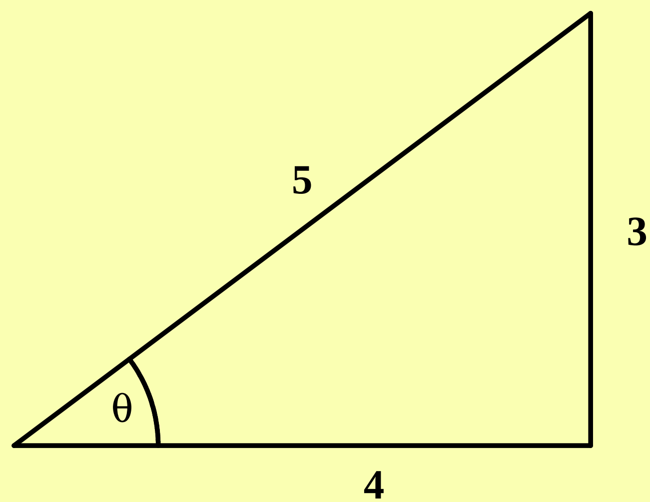
CIRCLE GEOMETRY





$r\theta = \text{arclength}$

$$\cos\theta = \frac{4}{5} \implies \tan\theta = \frac{3}{4}$$

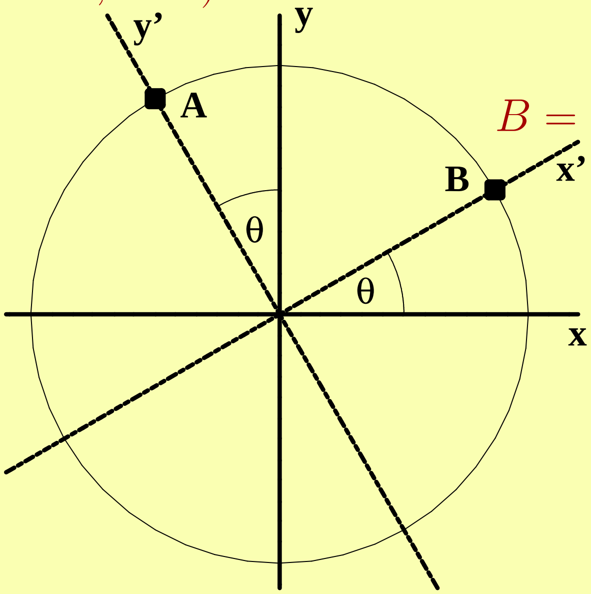
WHICH GEOMETRY?

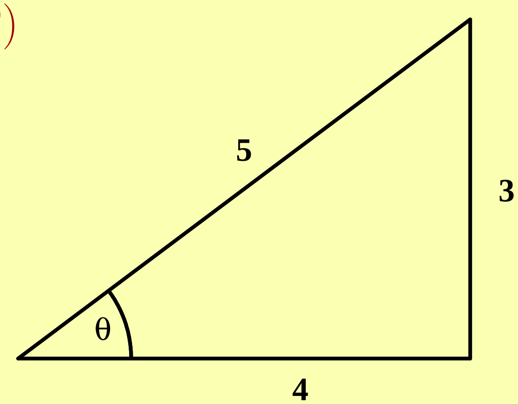
Euclidean

$$ds^2 = dx^2 + dy^2$$

$A = (-\sin\theta, \cos\theta)$

$B = (\cos\theta, \sin\theta)$



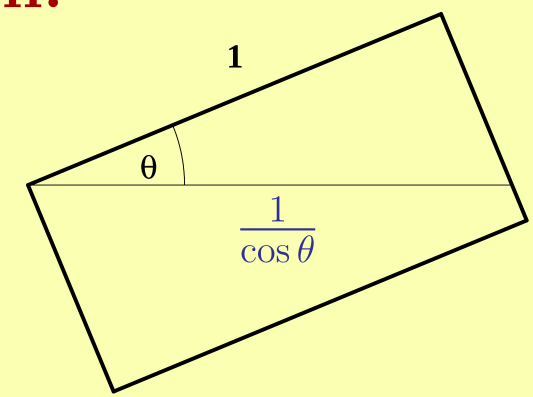


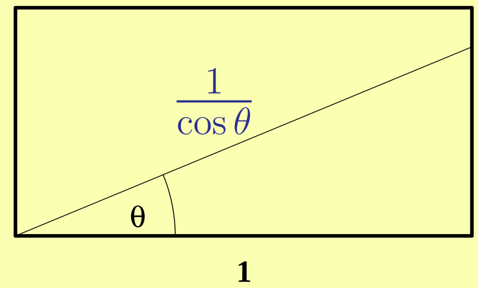
$$\tan\theta = \frac{3}{4}$$

Trigonometry!

MEASUREMENTS

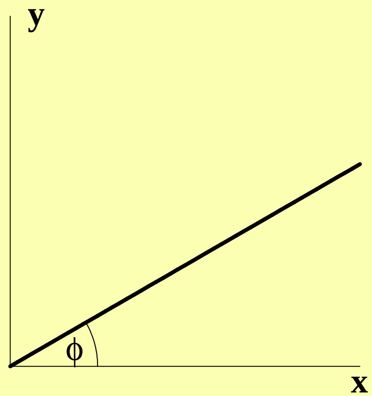
Width:

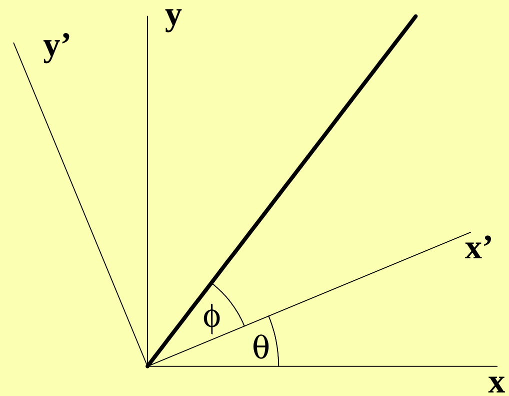




$$Apparent\ width > 1$$

Slope:



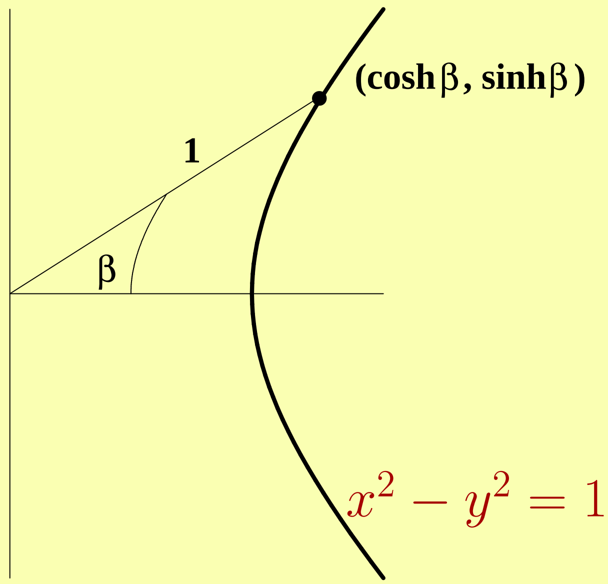


$$m \neq \tan\theta + \tan\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

Return

HYPERBOLA GEOMETRY



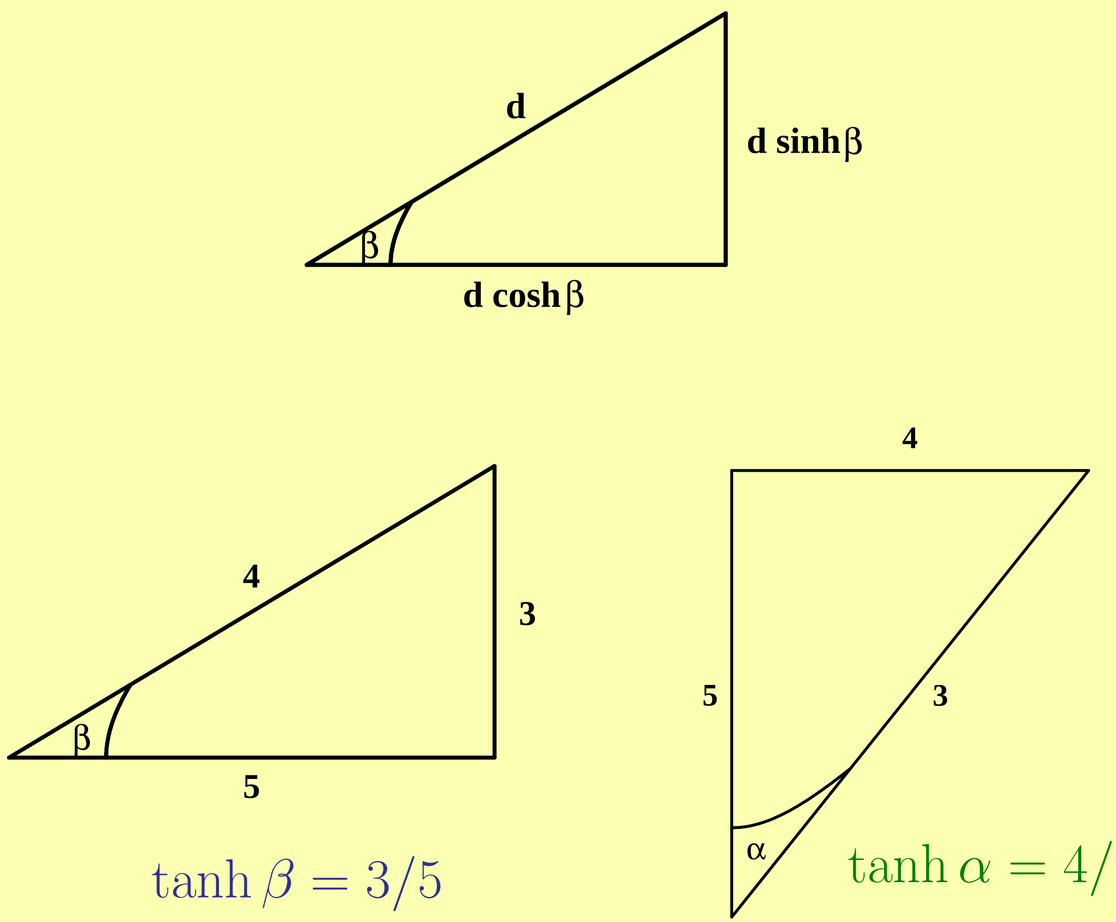
$$r\beta \;=\; \text{arclength}$$

$$ds^2 \;=\; |dx^2 - dy^2|$$

$$\cosh\beta \;=\; \frac{1}{2}\left(e^{\beta} + e^{-\beta}\right)$$

$$\sinh\beta \;=\; \frac{1}{2}\left(e^{\beta} - e^{-\beta}\right)$$

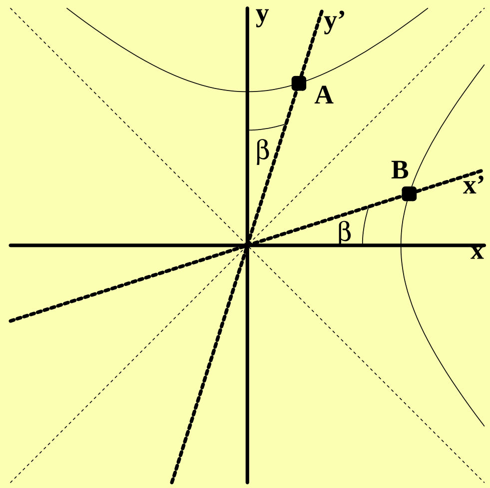
HYPERBOLIC TRIANGLE TRIG

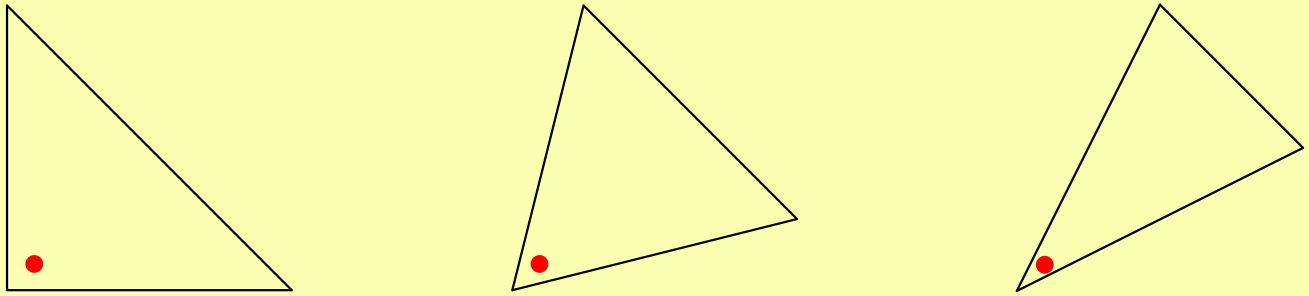


$\tanh \beta = 3/5$

$\tanh \alpha = 4/5$

RIGHT TRIANGLES



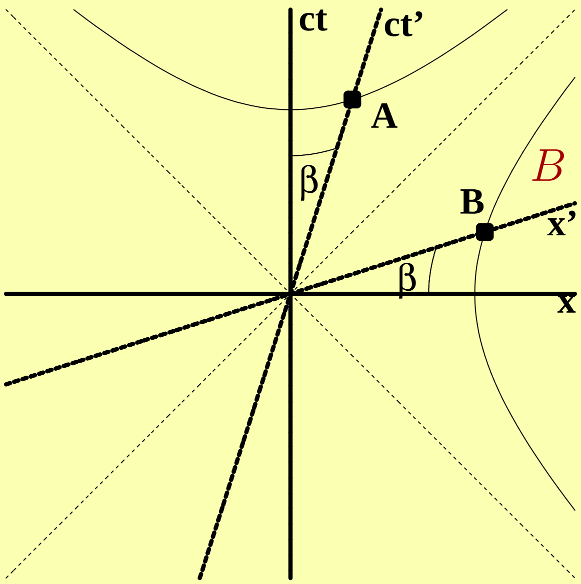


"right angles" are not angles!

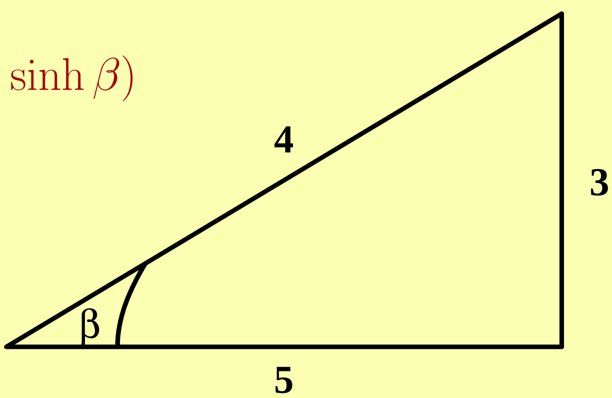
WHICH GEOMETRY?

signature	
$(+ + \ldots +)$	Euclidean
$(- + \ldots +)$	Minkowskian

$$ds^2 = -c^2\, dt^2 + dx^2$$



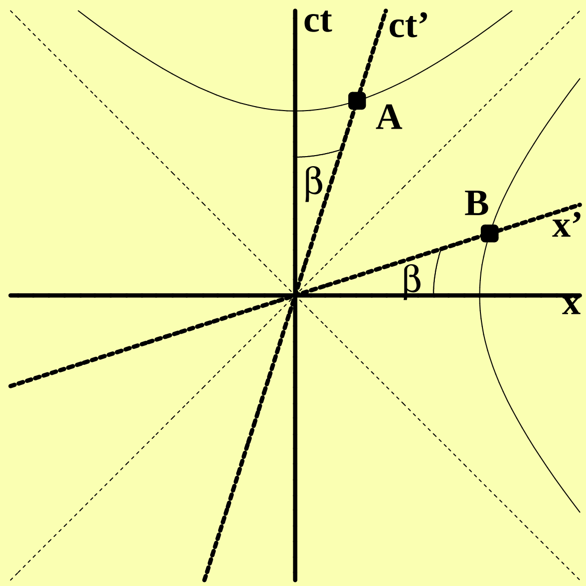
$B = (\cosh\beta, \sinh\beta)$



$\tanh\beta = \frac{3}{5}$

Special Relativity!

SPECIAL RELATIVITY



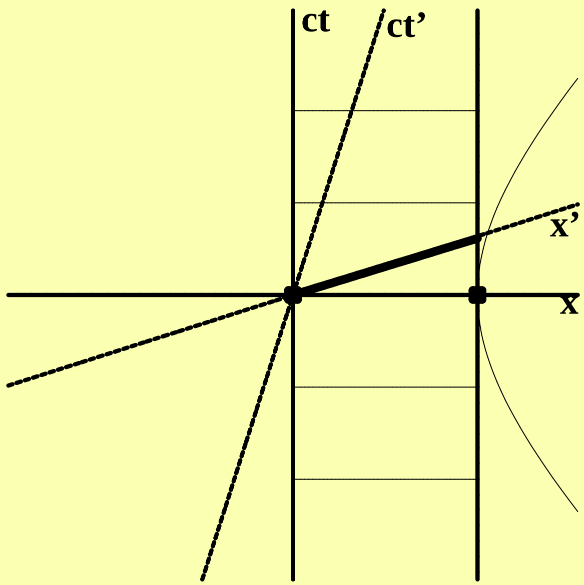
$$\frac{v}{c} = \tanh\beta$$

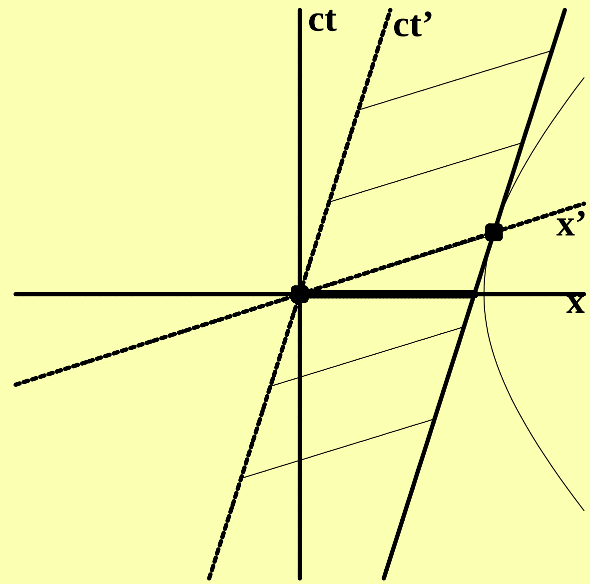
$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

Einstein addition formula!

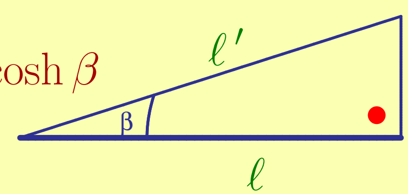
Compare

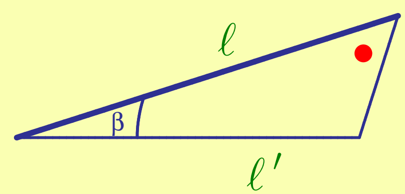
LENGTH CONTRACTION





$\ell = \ell' \cosh\beta$

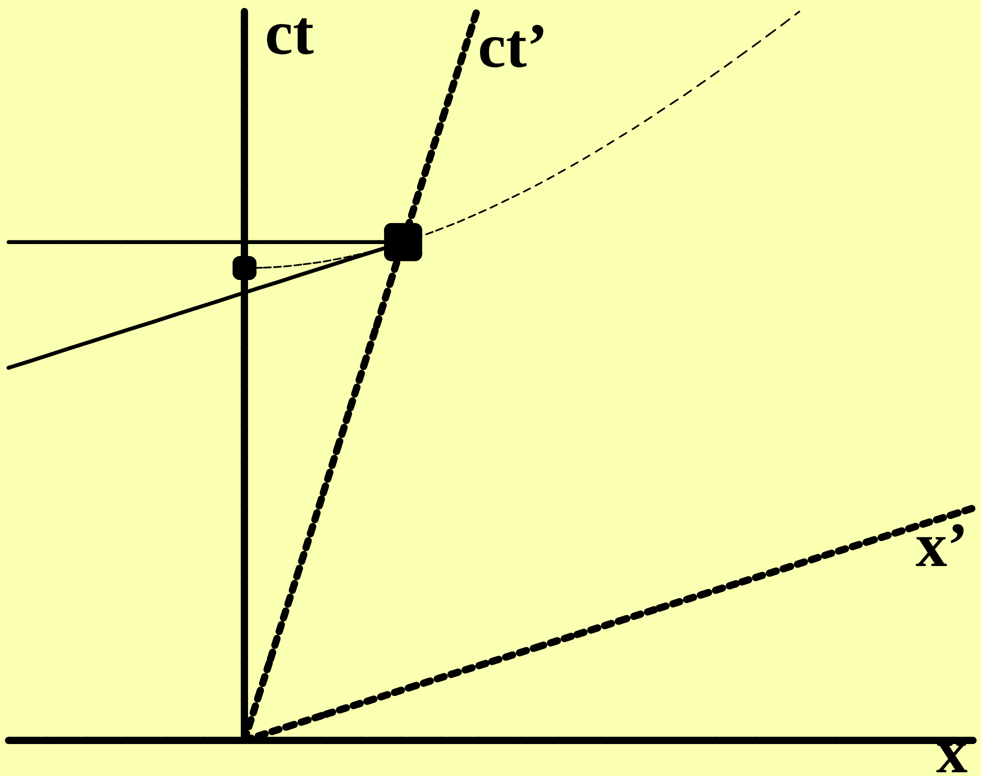




$$\ell' = \frac{\ell}{\cosh\beta}$$

Compare

TIME DILATION



COSMIC RAYS

*(Taylor & Wheeler, **1st edition**, Ex. 42, p. 89.)*

Consider μ-mesons produced by the collision of cosmic rays with gas nuclei in the atmosphere 60 kilometers above the surface of the earth, which then move vertically downward at nearly the speed of light. The half-life before μ-mesons decay into other particles is 1.5 microseconds (1.5×10^{-6} s).

- Assuming it doesn't decay, how long would it take a μ-meson to reach the surface of the earth?

$$\frac{60 \text{ km}}{3 \times 10^8 \, \frac{\text{m}}{\text{s}}} = 200 \ \mu\text{s}$$

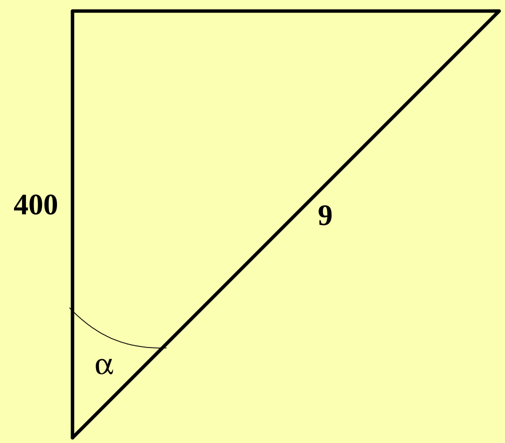
- Assuming there were no time dilation, about what fraction of the mesons reaches the earth?

$$\frac{200 \ \mu\text{s}}{\frac{3}{2} \ \mu\text{s per half-life}} = \frac{400}{3} \text{ half-lives}$$

- In actual fact, roughly $\frac{1}{8}$ of the mesons would reach the earth! How fast are they going?

COSMIC RAYS

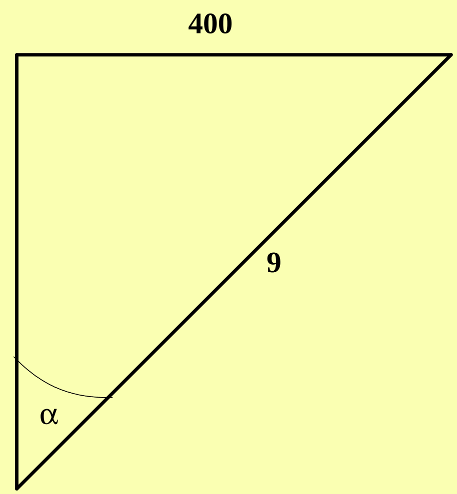
$$\frac{\frac{400}{3}\ \text{half-lives}}{3\ \ \text{half-lives}} = \frac{400}{9}$$



$$\frac{v}{c} = \tanh\alpha = \frac{\sqrt{400^2 - 9^2}}{400} \approx .99974684$$

COSMIC RAYS

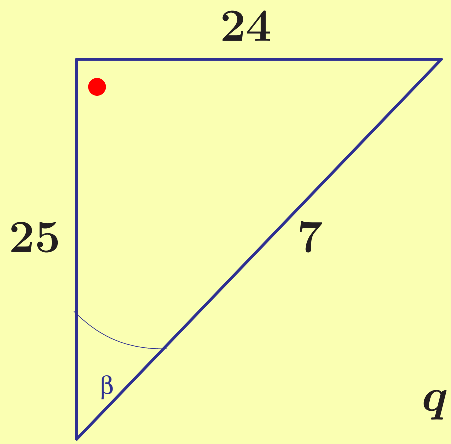
$$\frac{(60\ \mathrm{km})(1000\ \frac{\mathrm{m}}{\mathrm{km}})}{(4.5 \times 10^{-6}\ \mathrm{s})(3 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}})} = \frac{400}{9}$$



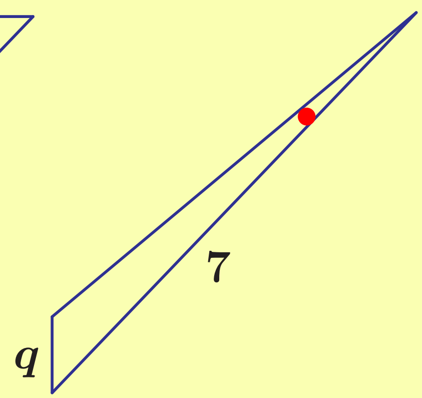
$$\frac{v}{c} = \tanh\alpha = \frac{400}{\sqrt{400^2 + 9^2}} \approx .99974697$$

TWIN PARADOX

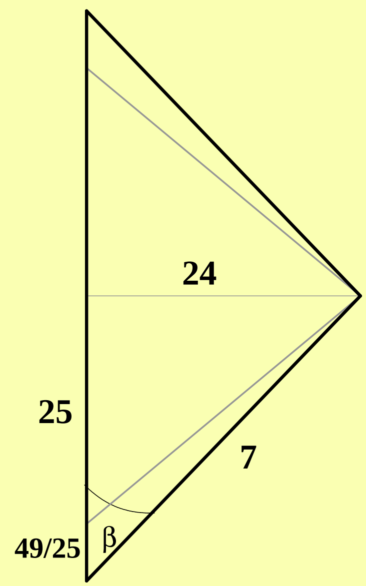
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$



$$q = \frac{7}{\cosh\beta} = \frac{49}{25}$$



Straight path takes longest!

CLASSICAL CONSERVATION LAWS

Momentum:

Lab Frame $$\sum m_i v_i = \sum \hat{m}_j \hat{v}_j$$

Moving Frame $$\sum m_i (v_i' + v) = \sum \hat{m}_j (\hat{v}_j' + v)$$

$$\boxed{\sum m_i v_i' = \sum \hat{m}_j \hat{v}_j' \iff \sum m_i = \sum \hat{m}_j}$$

Mass must be conserved!

Energy:

Lab Frame $$\frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum \hat{m}_j \hat{v}_j^2$$

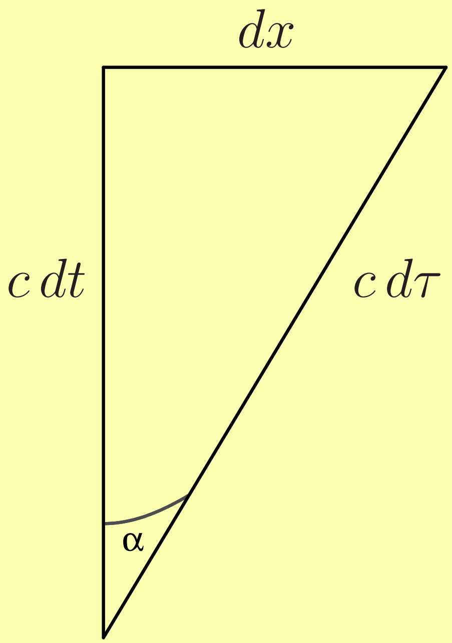
Moving Frame $$\frac{1}{2} \sum m_i (v_i' + v)^2 = \frac{1}{2} \sum \hat{m}_j (\hat{v}_j' + v)^2$$

Automatically conserved!

RELATIVISTIC MOMENTUM

$$u = \frac{dx}{dt} = c\tanh\alpha$$

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$



RELATIVISTIC CONSERVATION LAWS

Momentum: $(p = m\frac{dx}{d\tau} = mc\sinh\alpha)$

Lab Frame $$\sum m_i c\sinh\alpha_i = \sum \hat{m}_j c\sinh\hat{\alpha}_j$$

Moving Frame $$\sum m_i c\sinh(\alpha_i' + \beta) = \sum \hat{m}_j c\sinh(\hat{\alpha}_j' + \beta)$$

$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \Longrightarrow$

$$\begin{aligned} \sum m_i c\sinh\alpha_i' &= \sum \hat{m}_j c\sinh\hat{\alpha}_j' \\ \Longleftrightarrow \quad \sum m_i c\cosh\alpha_i' &= \sum \hat{m}_j c\cosh\hat{\alpha}_j' \\ \Longleftrightarrow \quad \sum m_i c\cosh\alpha_i &= \sum \hat{m}_j c\cosh\hat{\alpha}_j \end{aligned}$$

What is $mc\cosh\alpha$?

RELATIVISTIC CONSERVATION LAWS

$$\begin{aligned} mc\cosh\alpha &= mc\sqrt{1+\sinh^2\alpha} \\ &\approx mc\left(1+\frac{1}{2}\sinh^2\alpha\right) \end{aligned}$$

Energy:

$$\begin{aligned} E = mc^2\cosh\alpha &\approx mc^2 + \frac{p^2}{2m} \\ &\approx mc^2 + \frac{1}{2}mv^2 \end{aligned}$$

Momentum conserved $\Longleftrightarrow$ Energy conserved!

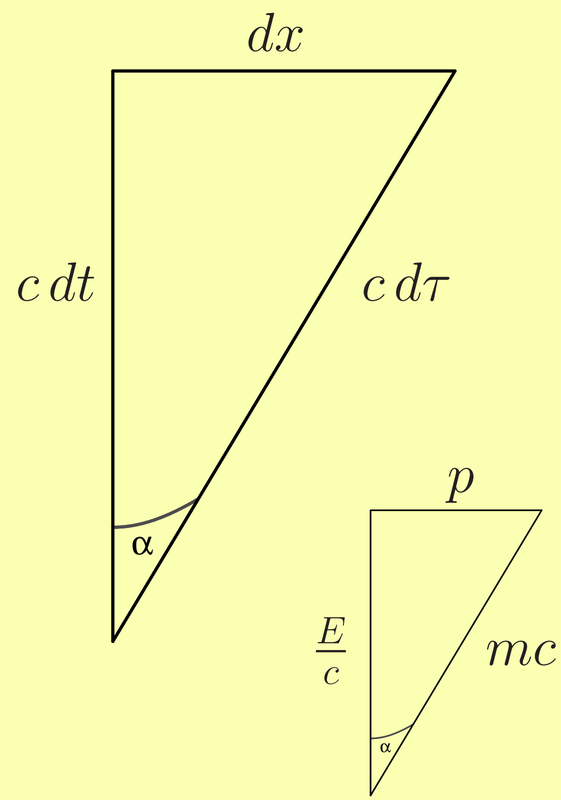
Mass need not be conserved!

RELATIVISTIC MOMENTUM

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$

$$E = mc^2\cosh\alpha = mc^2\frac{dt}{d\tau}$$

$$\begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = m\frac{d}{d\tau}\begin{pmatrix} ct \\ x \end{pmatrix}$$



ELECTROMAGNETISM

Capacitor at Rest: (charge density $\pm\sigma_0$)

$$\vec{\boldsymbol{E}}_0 = \frac{\sigma_0}{\epsilon_0}\,\hat{\boldsymbol{\jmath}}$$

$$\vec{\boldsymbol{B}}_0 = \vec{\boldsymbol{0}}$$

Moving Capacitor:

$$\begin{aligned}
\sigma &= \sigma_0 \cosh\alpha \\
\vec{\boldsymbol{\kappa}} &= \sigma\,\vec{\boldsymbol{u}} = \sigma c \tanh\alpha\,\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{E}} &= \frac{\sigma}{\epsilon_0}\,\hat{\boldsymbol{\jmath}} = K\cosh\alpha\,\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{B}} &= -\mu_0\vec{\boldsymbol{\kappa}} \times \hat{\boldsymbol{\jmath}} = -\tfrac{1}{c}\,K\sinh\alpha\,\hat{\boldsymbol{k}}
\end{aligned}$$

ELECTROMAGNETISM

Lab Frame:

$$\vec{\boldsymbol{E}} = K\cosh\alpha\,\hat{\boldsymbol{\jmath}}$$
$$\vec{\boldsymbol{B}} = -\tfrac{1}{c}\,K\sinh\alpha\,\hat{\boldsymbol{k}}$$

Moving Frame:

$$\vec{\boldsymbol{E}}' = K\cosh(\alpha+\beta)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{\boldsymbol{B}}' = -\tfrac{1}{c}\,K\sinh(\alpha+\beta)\,\hat{\boldsymbol{k}}$$

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta = \tfrac{E^y}{K}\cosh\beta - c\tfrac{B^z}{K}\alpha\sinh\beta$$
$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta = -c\tfrac{B^z}{K}\cosh\beta + \tfrac{E^y}{K}\sinh\beta$$

$$\Longrightarrow \boxed{\begin{aligned} E'^y &= E^y\cosh\beta - cB^z\sinh\beta \\ B'^z &= B^z\cosh\beta - \frac{1}{c}E^y\sinh\beta \end{aligned}}$$

LORENTZ TRANSFORMATIONS

$$\Lambda = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 0 & \frac{1}{c}E^x & \frac{1}{c}E^y & \frac{1}{c}E^z \\ -\frac{1}{c}E^x & 0 & B^z & -B^y \\ -\frac{1}{c}E^y & -B^z & 0 & B^x \\ -\frac{1}{c}E^z & B^y & -B^x & 0 \end{pmatrix}$$

$$\mathbf{v}' = \Lambda\mathbf{v}$$

$$\mathbf{F}' = \Lambda\mathbf{F}\Lambda^T$$

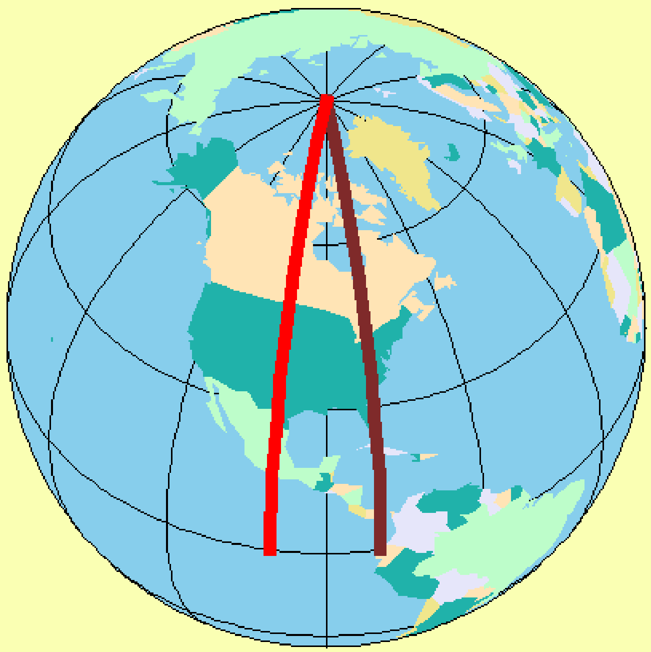
SUMMARY

- Lorentz transformations are hyperbolic rotations
- Beautifully treated in Taylor & Wheeler, 1st ed
- Removed from Taylor & Wheeler, 2nd edition
- Not currently covered in existing texts

http://www.math.oregonstate.edu/~tevian/geometry

WHICH GEOMETRY?

signature	flat	curved
$(+ + \dots +)$	Euclidean	Riemannian
$(- + \dots +)$	Minkowskian	



Tidal forces!

$$ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

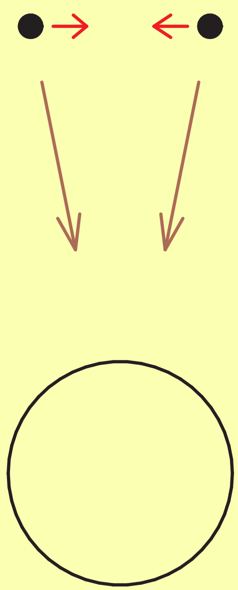
WHICH GEOMETRY?

signature	flat	curved
$(+ + \dots +)$	Euclidean	Riemannian
$(- + \dots +)$	Minkowskian	Lorentzian

$$ds^2 = -dt^2 + a(t)\, dx^2$$

Cosmology!

$(c = 1)$



$$ds^2 = -\left(1 - \tfrac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)}$$

$$+\, r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

General Relativity!

THE GEOMETRY OF SPECIAL RELATIVITY

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