Bridging the Gap between Mathematics and the Physical Sciences

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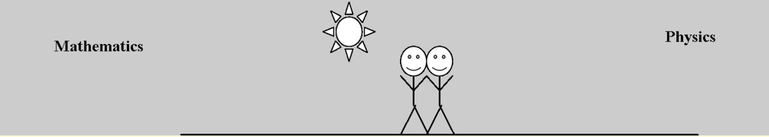
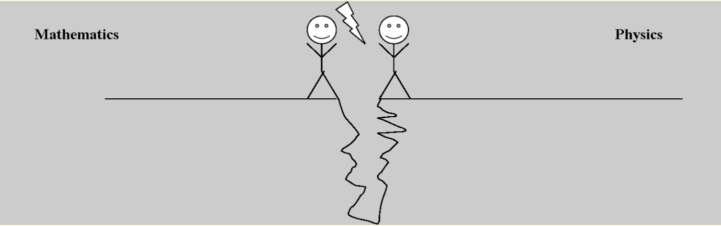
Mathematics vs. Physics



Mathematics vs. Physics

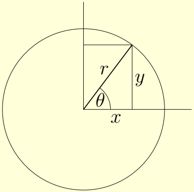


What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?



A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$

What are Functions?

MATH

$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

Two disciplines separated by a common language...

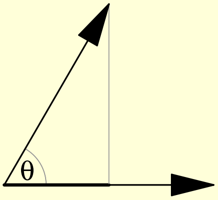
physical quantities $\neq$ functions

Mathematics vs. Physics

- **Physics is about things.**
- **Physicists can't change the problem.**

- **Mathematicians do algebra.**
- **Physicists do geometry.**

Write down something that you know about the dot product.

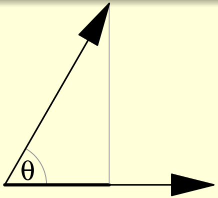


Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

Algebra:

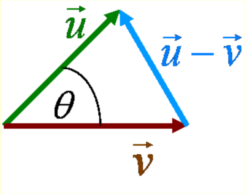
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

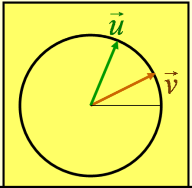
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



Law of Cosines:

$$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

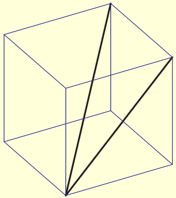


Addition Formulas:

$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \cos(\alpha - \beta)$$

Find the angle between the diagonal of a cube and the diagonal of one of its faces.



Algebra:

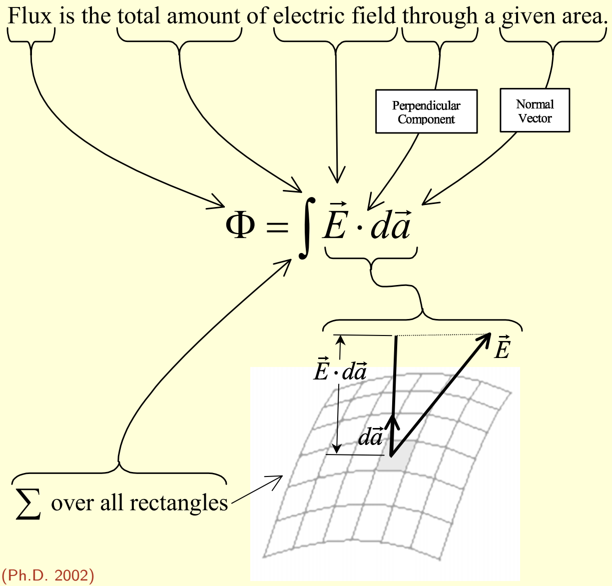
$$\vec{\mathbf{u}} = \hat{\imath} + \hat{\jmath} + \hat{k}$$
$$\vec{\mathbf{v}} = \hat{\imath} + \hat{k}$$

$$\Longrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2$$

Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta$$

Need both!



Kerry Browne (Ph.D. 2002)

CUPM

MAA Committee on the Undergraduate Program in Mathematics

Curriculum Guide

http://www.maa.org/cupm/cupm2004.pdf

CRAFTY

Subcommittee on Curriculum Renewal Across the First Two Years

Voices of the Partner Disciplines

http://www.maa.org/cupm/crafty

The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

- Small group activities
- Instructor's guide
- Online text (http://www.math.oregonstate.edu/BridgeBook)

http://www.math.oregonstate.edu/bridge

The Vector Calculus Bridge Project

Bridge Project homepage hits in 2009

Mathematicians' Line Integrals

- Start with Theory

$$\begin{aligned}\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, ds \\ &= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|} |\vec{\mathbf{r}}\,'(t)|\, dt \\ &= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt \\ &= \ldots = \int P\,dx + Q\,dy + R\,dz\end{aligned}$$

- Do examples starting from next-to-last line

Need parameterization $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$

Physicists' Line Integrals

- Theory
  - Chop up curve into little pieces $d\vec{\mathbf{r}}$.
  - Add up components of $\vec{\mathbf{F}}$ parallel to curve (times length of $d\vec{\mathbf{r}}$)
- Do examples directly from $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Need $d\vec{\mathbf{r}}$ along curve

Mathematics

$$\vec{\mathbf{F}}(x,y) = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2} \qquad \vec{\mathbf{r}} = x\,\hat{\imath} + y\,\hat{\jmath}$$

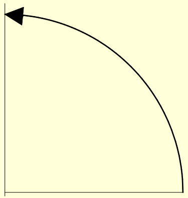
$$\begin{aligned} x &= 2\cos\theta \\ y &= 2\sin\theta \end{aligned}$$

$$\begin{aligned} \int \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_0^{\frac{\pi}{2}} \vec{\mathbf{F}}(x(\theta),y(\theta))\cdot\vec{\mathbf{r}}\,'(x(\theta),y(\theta))\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\sin\theta\,\hat{\imath}+\cos\theta\,\hat{\jmath})\cdot 2(-\sin\theta\,\hat{\imath}+\cos\theta\,\hat{\jmath})\,d\theta \\ &= \ldots = \frac{\pi}{2} \end{aligned}$$

Physics

$$\vec{\mathbf{F}} = \frac{\hat{\phi}}{r}$$

$$d\vec{\mathbf{r}} = r\,d\phi\,\hat{\phi}$$



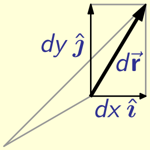
I: $|\vec{\mathbf{F}}| = \text{const}; \; \vec{\mathbf{F}} \parallel d\vec{\mathbf{r}} \Longrightarrow$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{1}{2}(2\;\frac{\pi}{2})$$

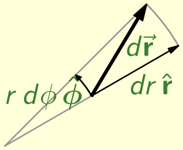
II: Do the dot product $\longmapsto$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\frac{\pi}{2}} \frac{\hat{\phi}}{2} \cdot 2\,d\phi\,\hat{\phi} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2}$$

Vector Differentials



$$d\vec{\mathbf{r}} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

Gradient

Master Formula:

$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$$f = \text{const} \Longrightarrow df = 0 \quad \Longrightarrow \quad \vec{\nabla} f \perp d\vec{\mathbf{r}}$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{d\vec{\mathbf{r}}}{|d\vec{\mathbf{r}}|}$$

Gradient

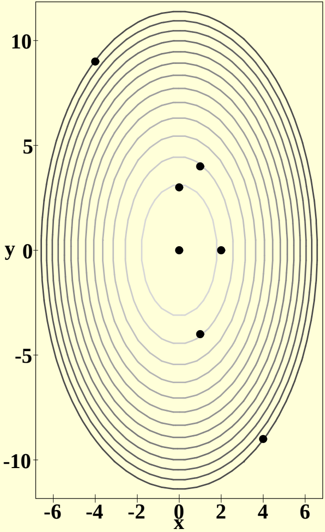
The gradient of a function is a vector field that points in the direction in which the function increases most rapidly, and whose magnitude is the amount of that increase.

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

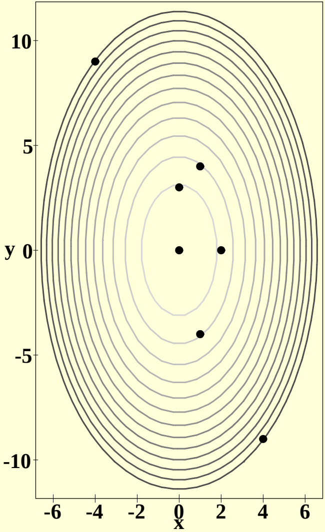
where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



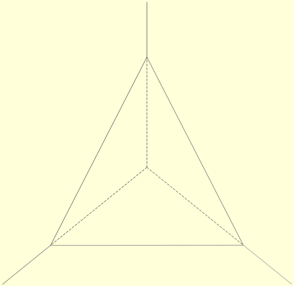
The Hill

Stand up and close your eyes.

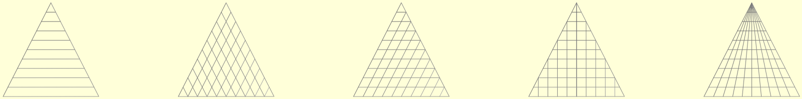
Hold out your right arm in the direction of the gradient where you are standing.



What is the flux of the vector field $\vec{\mathbf{E}} = z\hat{k}$ upwards through the triangular region connecting the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$?



First decide how to chop up the region:



Chop parallel to the x and y axes:

$$d\vec{\mathbf{r}} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

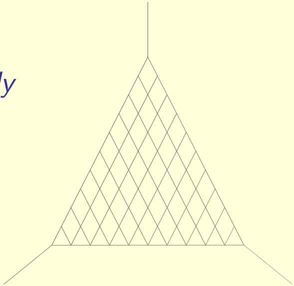
$\{x + y + z = 1\} \Longrightarrow$

Use what you know!

$$\begin{aligned} d\vec{\mathbf{r}}_1 &= (\,\hat{\imath} - \hat{\jmath}\,)\,dx \qquad (y = \text{const}) \\ d\vec{\mathbf{r}}_2 &= (\,\hat{\jmath} - \hat{k}\,)\,dy \qquad (x = \text{const}) \\ \Longrightarrow d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 = (\,\hat{\imath} + \hat{\jmath} + \hat{k}\,)\,dx\,dy \end{aligned}$$

$\vec{\mathbf{E}} = z\hat{k} \Longrightarrow$

$$\int_T \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_0^1 \int_0^{1-y} (1-x-y)\,dx\,dy = \frac{1}{6}$$



Coherent Calculus

co-he-rent:

logically or aesthetically ordered

cal-cu-lus:

a method of computation *in a special notation*

differential calculus:

a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*

Differentials

$$
\begin{aligned}
d(u + cv) &= du + c\,dv \\
d(uv) &= u\,dv + v\,du
\end{aligned}
$$

$$
\begin{aligned}
d\left(u^n\right) &= nu^{n-1}\,du \\
d\left(e^u\right) &= e^u\,du \\
d(\sin u) &= \cos u\,du \\
d(\cos u) &= -\sin u\,du \\
d(\ln u) &= \frac{1}{u}\,du
\end{aligned}
$$

Derivatives

functions

Derivatives:

$$f(x) = \sin(x) \Longrightarrow f'(x) = \cos(x)$$

Chain rule:

$$h(x) = f\big(g(x)\big) \Longrightarrow h'(x) = f'\big(g(x)\big)g'(x)$$

Inverse functions:

$$g(x) = f^{-1}(x) \Longrightarrow g'(x) = \frac{1}{f'\big(g(x)\big)}$$

Derivatives

physical quantities $\neq$ functions

Derivatives:

$$\frac{d}{du}\sin u = \frac{d\sin u}{du} = \cos u$$

Chain rule:

$$\frac{d}{dx}\sin u = \frac{d\sin u}{dx} = \frac{d\sin u}{du}\,\frac{du}{dx} = \cos u\,\frac{du}{dx}$$

Inverse functions:

$$\frac{d}{du}\ln u = \frac{d}{du}\,q = \frac{dq}{du} = \frac{1}{du/dq} = \frac{1}{de^q/dq} = \frac{1}{e^q} = \frac{1}{u}$$

Derivatives

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with d"

Tevian Dray & Corinne A. Manogue,
Putting Differentials Back into Calculus,
College Math. J. **41**, 90–100 (2010).

A Radical View of Calculus

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.

- The central idea in calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is "use what you know".
- The central representation of a function is data attached to the domain.

SUMMARY

I took this class a year ago, and I still remember all of it...