The Geometry of Relativity, & Piecewise Conserved Quantities

Tevian Dray

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Personal History

- **1987:** Indo-American Fellow @ IMSc, TIFR
  (visits to Pune and RRI)
- **1988:** Returned to RRI and TIFR
- Collaborated with Sam:
  Tevian Dray, Ravi Kulkarni, and Joseph Samuel,
  Duality and Conformal Structure,
  J. Math. Phys. **30**, 1306–1309 (1989).

Differential Geometry

Definition

A *topological manifold* is a second countable Hausdorff space that is locally homeomorphic to Euclidean space. A *differentiable manifold* is a topological manifold equipped with an equivalence class of atlases whose transition maps are differentiable.

General Relativity $\neq$ Differential Geometry

What math is needed for GR??

Background

- Differential geometry course: Rick Schoen
- GR reading course: MTW
- GR course: Sachs–Wu
- designed and taught undergrad math course in GR: Schutz, d'Inverno, Wald, Taylor–Wheeler, Hartle
- designed and taught undergrad physics course in SR
- NSF-funded curricular work (math and physics) since 1996
- national expert in teaching 2nd-year calculus
  http://blogs.ams.org/matheducation

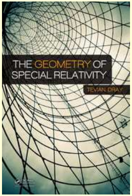
geometer, relativist, curriculum developer, education researcher
Mathematics, Physics, PER, RUME

Mathematics vs. Physics

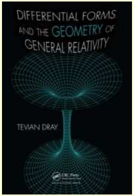
My mathematics colleagues think I'm a physicist.

My physics colleagues know better.

Books

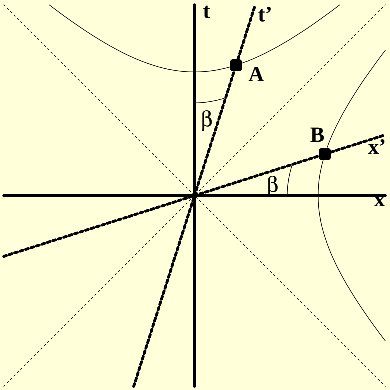


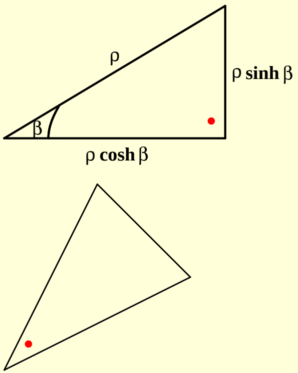
The Geometry of Special Relativity
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
http://relativity.geometryof.org/GSR



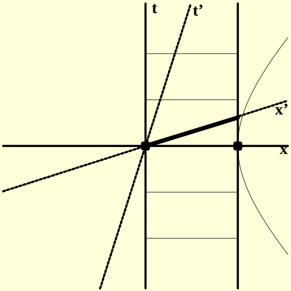
Differential Forms and the Geometry of General Relativity
Tevian Dray
A K Peters/CRC Press 2014
ISBN: 978-1-4665-1000-5
http://relativity.geometryof.org/GDF
http://relativity.geometryof.org/GGR

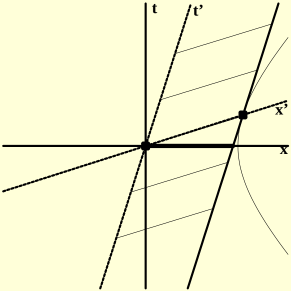
Trigonometry



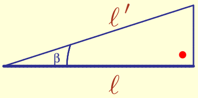


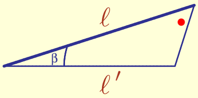
Length Contraction





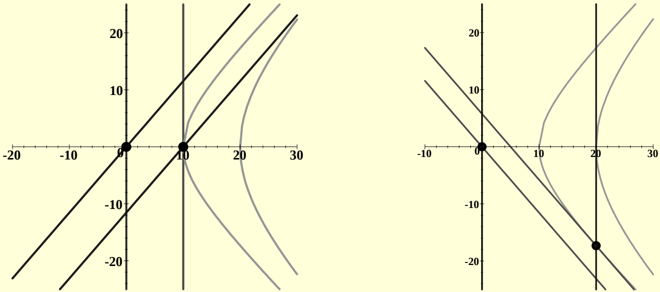
$$\ell' = \frac{\ell}{\cosh\beta}$$





Paradoxes

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn?

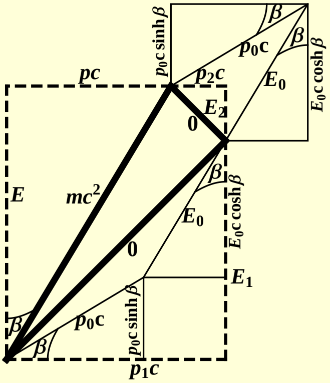


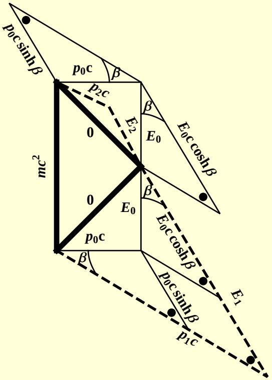
barn frame

pole frame

Relativistic Mechanics

A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ *decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon.* $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$





Addition Formulas

$$v = c \tanh \beta$$

Einstein Addition Formula:

$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta} \qquad \left(\text{“}v + w\text{''} = \frac{v+w}{1+vw/c^2}\right)$$

Conservation of Energy-Momentum:

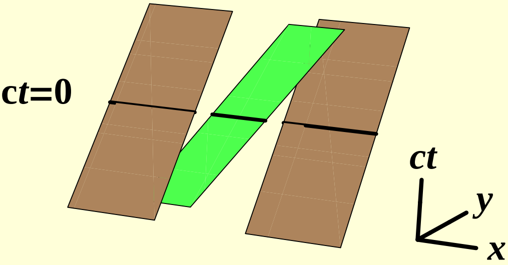
$$p = mc \sinh \alpha$$
$$E = mc^2 \cosh \alpha$$

Moving Capacitor:

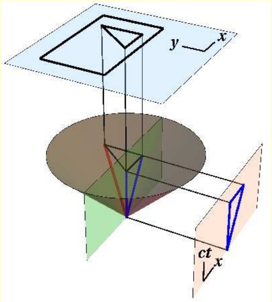
$$E'^y = C \cosh(\alpha + \beta) = E^y \cosh \beta - cB^z \sinh \beta$$
$$cB'^z = -C \sinh(\alpha + \beta) = cB^z \cosh \beta - E^y \sinh \beta$$

3d spacetime diagrams

(rising manhole)



$$(v\,\Delta t)^2 + (c\,\Delta t')^2 = (c\,\Delta t)^2$$

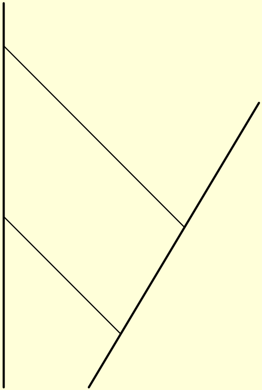


$$(v\,\Delta t)^2 - (c\,\Delta t)^2 = -(c\,\Delta t')^2$$

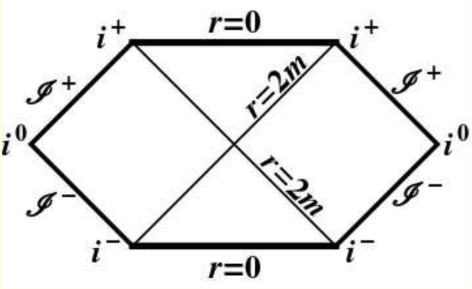
http://relativity.geometryof.org/GSR/book/updates/3d

Am. J. Phys. **81**, 593–596 (2013)

The Geometry of General Relativity

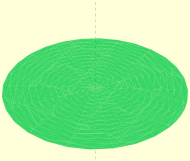


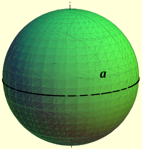
Doppler effect (SR)
Cosmological redshift (GR)

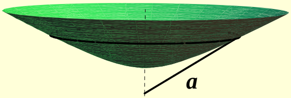


Asymptotic structure

Line Elements







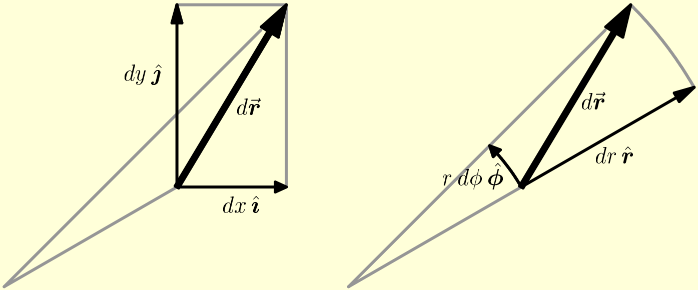
$dr^2 + r^2\, d\phi^2$

$d\theta^2 + \sin^2\theta\, d\phi^2$

$d\beta^2 + \sinh^2\beta\, d\phi^2$

Vector Calculus

$$ds^2 = d\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}}$$



$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

Differential forms are integrands: $(*^2 = 1)$

$$f = f \qquad \text{(0-form)}$$

$$F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \qquad \text{(1-form)}$$

$$*F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} \qquad \text{(2-form)}$$

$$*f = f\,dV \qquad \text{(3-form)}$$

Exterior derivative: $(d^2 = 0)$

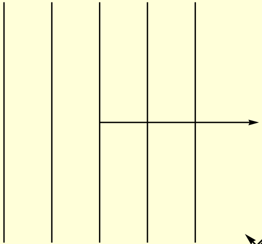
$$df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}$$

$$dF = \vec{\nabla} \times \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}$$

$$d{*F} = \vec{\nabla} \cdot \vec{\boldsymbol{F}}\,dV$$

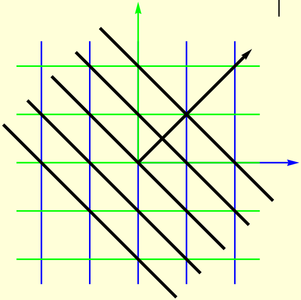
$$d{*f} = 0$$

The Geometry of Differential Forms

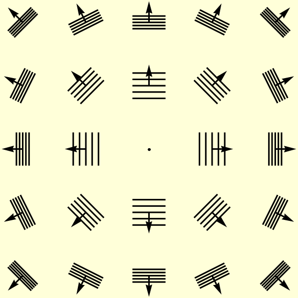


dx

$dx + dy$



$r\,dr = x\,dx + y\,dy$



Geodesic Equation

Orthonormal basis: $d\vec{\boldsymbol{r}} = \sigma^i\, \hat{\mathbf{e}}_i$ $\quad (\Longrightarrow ds^2 = d\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}})$

Connection:

$$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{\boldsymbol{v}}\, d\lambda = d\vec{\boldsymbol{r}}$$

$$\dot{\vec{\boldsymbol{v}}} = 0$$

Symmetry:

$$d\vec{\boldsymbol{X}} \cdot d\vec{\boldsymbol{r}} = 0$$

$$\Longrightarrow \vec{\boldsymbol{X}} \cdot \vec{\boldsymbol{v}} = \text{const}$$

Example: Polar Coordinates

Symmetry:

$$ds^2 = dr^2 + r^2\, d\phi^2$$

$$\Longrightarrow d\vec{\boldsymbol{r}} = dr\, \hat{\boldsymbol{r}} + r\, d\phi\, \hat{\boldsymbol{\phi}}$$

$$\Longrightarrow r\, \hat{\boldsymbol{\phi}} \text{ is Killing}$$

Idea: $df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} \Longrightarrow r\, \hat{\boldsymbol{\phi}} \cdot \vec{\nabla} f = \frac{\partial f}{\partial \phi} \Longrightarrow r\, \hat{\boldsymbol{\phi}} = \frac{\partial}{\partial \phi}$

Check: $d(r\, \hat{\boldsymbol{\phi}}) = dr\, \hat{\boldsymbol{\phi}} + r\, d\hat{\boldsymbol{\phi}} = dr\, \hat{\boldsymbol{\phi}} - r\, d\phi\, \hat{\boldsymbol{r}} \perp d\vec{\boldsymbol{r}}$

Geodesic Equation:

$$\vec{\boldsymbol{v}} = \dot{r}\, \hat{\boldsymbol{r}} + r\, \dot{\phi}\, \hat{\boldsymbol{\phi}} \Longrightarrow r\, \hat{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{v}} = r^2\, \dot{\phi} = \ell$$

$$\Longrightarrow 1 = \dot{r}^2 + r^2\, \dot{\phi}^2 = \dot{r}^2 + \frac{\ell^2}{r^2}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\boldsymbol{G}} = G^i\,\hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j\,\hat{\mathbf{e}}_i$$

$$\Longrightarrow d{*}\vec{\boldsymbol{G}} = 0$$

Field equation:

$$\boxed{\vec{\boldsymbol{G}} + \Lambda\,d\vec{\boldsymbol{r}} = 8\pi\,\vec{\boldsymbol{T}}}$$

(vector valued 1-forms, not tensors)

Stress-Energy Tensor

$$d\vec{r} = \sigma^a \,\hat{e}_a$$

Vector-valued 1-form:

$$\vec{T} = T^a{}_b\, \sigma^b \,\hat{e}_a$$

3-form:

$$\tau^a = *T^a$$

Conservation:

$$d\left(\tau^a \,\hat{e}_a\right) = 0$$

$$*d*\vec{T} = 0$$

What about Tensors?

What tensors are needed to do GR?

Metric? Use $d\vec{r}$! (vector-valued 1-form!)
Curvature? Riemann tensor is really a 2-form. (Cartan!)
Ricci? Einstein? Stress-Energy? Vector-valued 1-forms!

$\exists$ only 1 essential symmetric tensor in GR!

Killing eq: $d\vec{X} \cdot d\vec{r} = 0$

Students understand line elements...

$$ds^2 = d\vec{r} \cdot d\vec{r}$$

Topic Order

Examples First!

- Schwarzschild geometry can be analyzed using vector calculus.
- Rain coordinates! (Painlevé-Gullstrand; freely falling)

Geodesics:

- EBH: Principle of Extremal Aging
- Hartle: variational principle (Lagrangian mechanics?)
- TD: "differential forms without differential forms"

Choices

Language:

- Mathematicians: invariant objects (no indices)
- Physicists: components (indices)
- Relativists: abstract index notation ("indices without indices")
- Cartan: curvature without tensors

$\therefore$ use differential forms?

Coordinates:

- Mathematicians: coordinate basis (usually)
- Physicists: calculate in coordinates; interpret in orthonormal basis
- Equivalence problem: 79310 coordinate components reduce to 8690

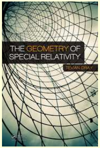
$\therefore$ use orthonormal frames? ($d\vec{r}$?)

Does it work?

- I am a mathematician...
- There is no GR course in physics department.
  (I developed the SR course.)
- Core audience is undergraduate math and physics majors.
  (Many double majors.)
- Hartle's book:
  Perfect for physics students, but tough for math majors.
- My course: 10 weeks differential forms, then 10 weeks GR.
  (Some physics students take only GR, after "crash course".)

In this context: **YES!**

SUMMARY #1



http://relativity.geometryof.org/GSR
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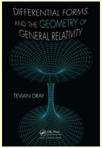


- Special relativity is hyperbolic trigonometry!
- General relativity can be described without tensors!
- BUT: Need vector-valued differential forms...

Shells

- **Dray & 't Hooft (1985)**
  Gravitational shock wave of a massless particle;
  Shells of matter at horizon of Schwarzschild black hole
- **Clarke & Dray (1987)**
  Junction conditions for null hypersurfaces
- **Dray & Padmanabhan (1988)**
  Conserved quantities from piecewise Killing vectors
- **Dray & Joshi (1990)**
  Glueing Reissner-Nordstrøm spacetimes together
  Reissner–Nordström

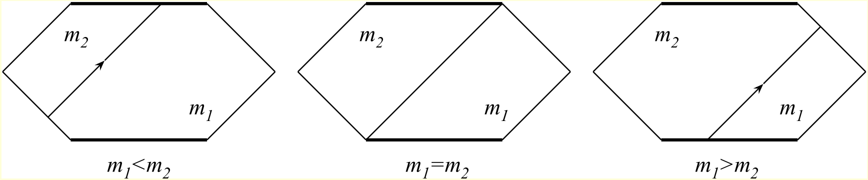
  ⋮
- **Hazboun & Dray (2010)**
  Negative-energy shells in Schwarzschild spacetimes

Signature Change

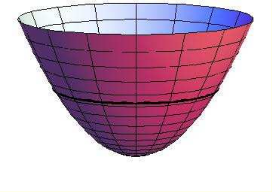
- **Dray, Manogue, & Tucker (1991); Ellis et al. (1992)**
  Scalar field in the presence of signature change
- **Dray & Hellaby (1994); Hellaby & Dray (1994)**
  Patchwork Divergence Theorem
- **Schray, Dray, Manogue, Tucker, & Wang (1996)**
  Spinors and signature change
- **Dray (1996); Dray, Ellis, Hellaby, & Manogue (1997)**
  Gravity and signature change
- **Dray (1997); Hartley, Tucker, Tuckey, & Dray (2000)**
  Tensor distributions in the presence of signature change

Gluing Spacetimes Together

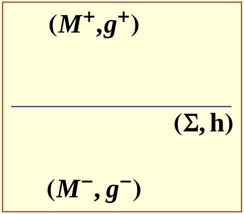
Shells:



Signature Change:



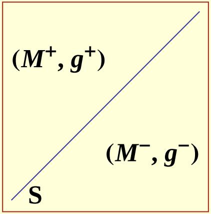
Spacelike Boundaries



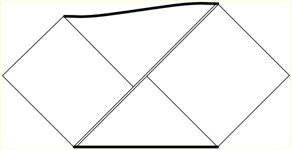
$$g_{ab} = (1-\Theta)\, g^-_{ab} + \Theta\, g^+_{ab}$$
$$[g_{ab}] = g^+_{ab}\big|_\Sigma - g^-_{ab}\big|_\Sigma = 0$$

$$\Longrightarrow R_{ab} = (1-\Theta)\, R^-_{ab} + \Theta\, R^+_{ab} + \delta_c[\Gamma^c{}_{ab}] - \delta_b[\Gamma^c{}_{ac}]$$
$$= (1-\Theta)\, R^-_{ab} + \Theta\, R^+_{ab} + \delta\rho_{ab}$$
$$(\delta_c = \delta n_c = \nabla_c \Theta)$$

Null Boundaries

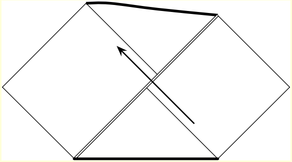


Dray & 't Hooft (1996)



$m > 0$

Hazboun & Dray (2010)



$m < 0$

Conserved Quantities

Piecewise Killing vector:

$$\xi^a = (1-\Theta)\,\xi^a_- + \Theta\,\xi^a_+$$
$$\Longrightarrow \nabla_{(a}\xi_{b)} = \left[\xi_{(a}\right]\delta_{b)}$$

Darmois junction conditions: $([h_{ij}] = 0 = [K^{ij}])$

$$\Longrightarrow [T^{ab}] = 0$$
$$\Longrightarrow \nabla_a(T^{ab}\xi_b) = (\nabla_a T^{ab})\xi_b + T^{ab}\nabla_a\xi_b$$
$$= 0 + T^{ab}\,[\xi_a]\,\delta_b$$

$\therefore$ conserved quantity if $[\xi_a] = \Xi n_a$ & $T^{ab}n_a n_b = 0$

Patchwork Divergence Theorem

Divergence Theorem, X smooth: $(m \wedge \sigma = \omega)$

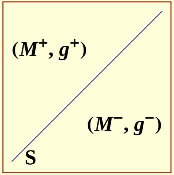
$$\mathrm{div}(X)\,\omega = \mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega)$$

$$\Longrightarrow \int_W \mathrm{div}(X)\,\omega = \oint_{\partial W} i_X\omega = \oint_{\partial W} m(X)\,\sigma$$

X piecewise smooth: (m_0 from M^- to M^+)

$$\int_W \mathrm{div}(X)\,\omega = \oint_{\partial W} m(X)\,\sigma - \int_\Sigma m_0([X])\,\sigma^0$$

Shells



Two Schwarzschild regions:

$$ds^2 = \begin{cases} -\frac{32m^3}{r} e^{-r/2m} du\, dv + r^2\, d\Omega^2 & (u \le \alpha) \\ -\frac{32m^3}{r} e^{-r/2M} dU\, dV + r^2\, d\Omega^2 & (u \ge \alpha) \end{cases}$$

$$[g] = 0 \Longrightarrow \frac{\alpha}{m} = \frac{U(\alpha)}{MU'(\alpha)} \Longrightarrow \frac{u\partial_u}{m} = \frac{U\partial_U}{M}$$

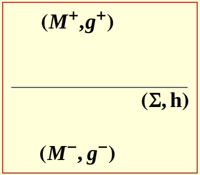
$$\xi = (1 - \Theta)\, \frac{v\partial_v - u\partial_u}{4m} + \Theta\, \frac{V\partial_V - U\partial_U}{4M} \Longrightarrow [\xi] \sim \partial_V$$

$$T_{uu} = \frac{\delta}{\gamma\pi r^2}\, (M - m) \Longrightarrow T_{vv} = 0$$

Conserved quantity:

$$-\int_\Sigma \left((1 - \Theta)\; T^t{}_t + \Theta\; T^T{}_T \right) dS = M - m$$

Signature Change



$$ds^2 = \begin{cases} +dt^2 + h_{ij}\, dx^i\, dx^j & (t < 0) \\ -dt^2 + h_{ij}\, dx^i\, dx^j & (t > 0) \end{cases}$$

Volume element is continuous!!

$$\rho := G_{ab} n^a n^b = \frac{1}{2}\Big((K^c{}_c)^2 - K_{ab}K^{ab} - \epsilon R \Big)$$

$$\text{Darmois} \Longrightarrow [R] = 0 = [K_{ab}] \text{ but } [\epsilon] \neq 0$$

$$\Longrightarrow [\rho] = -R \neq 0$$

"Energy density at change of signature"

Geometric Reasoning

Iedereen in deze kamer kan twee talen praten
(Everyone in this room is bilingual.)

Mathematics $\neq$ Physics

Mathematicians teach algebra;
Physicists do geometry!

Geometric Reasoning

Vector Calculus Bridge Project:

http://math.oregonstate.edu/bridge

- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (http://math.oregonstate.edu/BridgeBook)

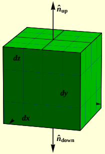
Paradigms in Physics Project:

http://physics.oregonstate.edu/portfolioswiki

- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*

Lorentzian Vector Calculus

Minkowski 3-space:



$$\hat{x}\cdot\hat{x} = 1 = \hat{y}\cdot\hat{y}, \qquad \hat{t}\cdot\hat{t} = -1$$

$$\vec{F} = F^x\,\hat{x} + F^y\,\hat{y} + F^t\,\hat{t}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F^x}{\partial x} + \frac{\partial F^y}{\partial y} + \frac{\partial F^t}{\partial t};$$

Divergence Theorem:

$$\int\!\!\int\!\!\int_W \vec{\nabla}\cdot\vec{F}\;dx\,dy\,dt = \int_{\partial W} \vec{F}\cdot\hat{n}\;dA$$

where $\hat{n}$ = **outward** normal in spacelike directions,
but $\hat{n}$ = **inward** normal in timelike directions!

SUMMARY

- Junction conditions at null boundary are surprising!
- Junction conditions at signature change are surprising!
- The Divergence Theorem in Minkowski space is surprising!
- All of these results follow from Patchwork Divergence Thm.

THANK YOU

http://oregonstate.edu/~drayt/talks/RRI17.pdf

http://relativity.geometryof.org/GSR/bookinfo
http://relativity.geometryof.org/DFGGR/bookinfo
https://arxiv.org/abs/1701.02863