The GEOMETRY of the OCTONIONS

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I: Octonions
II: Rotations
III: Lorentz Transformations
IV: Dirac Equation
V: Exceptional Groups
VI: Eigenvectors

DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$

$$z = x + yi$$

Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$

$$q = (a + bi) + (c + di)j$$

Octonions:

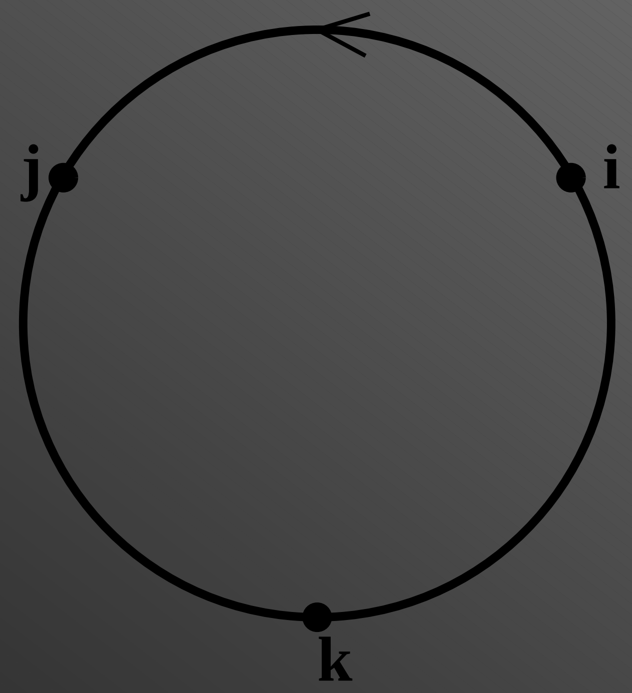
$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

$$i^2 = j^2 = \ell^2 = -1$$

QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

not commutative



VECTORS I

$$v = bi + cj + dk \longleftrightarrow \vec{\boldsymbol{v}} = b\,\hat{\boldsymbol{\imath}} + c\,\hat{\boldsymbol{\jmath}} + d\,\hat{\boldsymbol{k}}$$

$$vw \longleftrightarrow -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$$

Dot product exists in any dimension

but

Cross product exists only in 3 and 7 dimensions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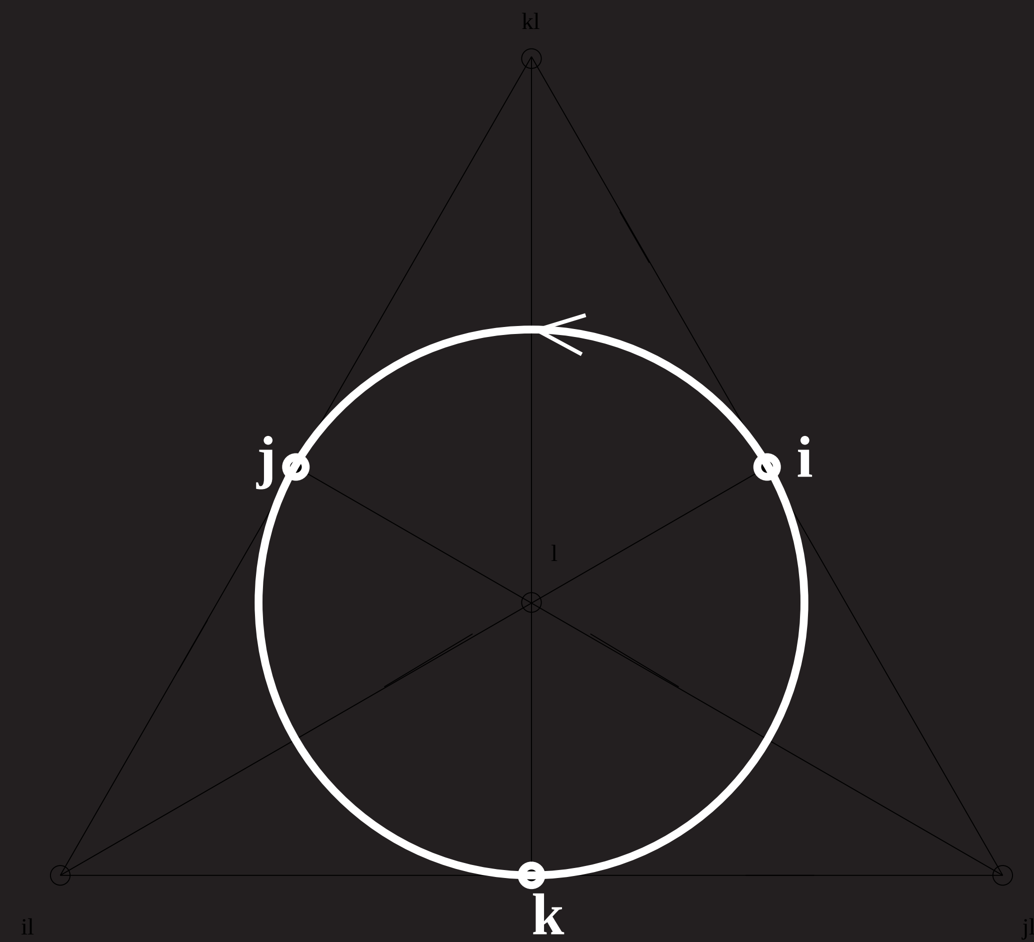

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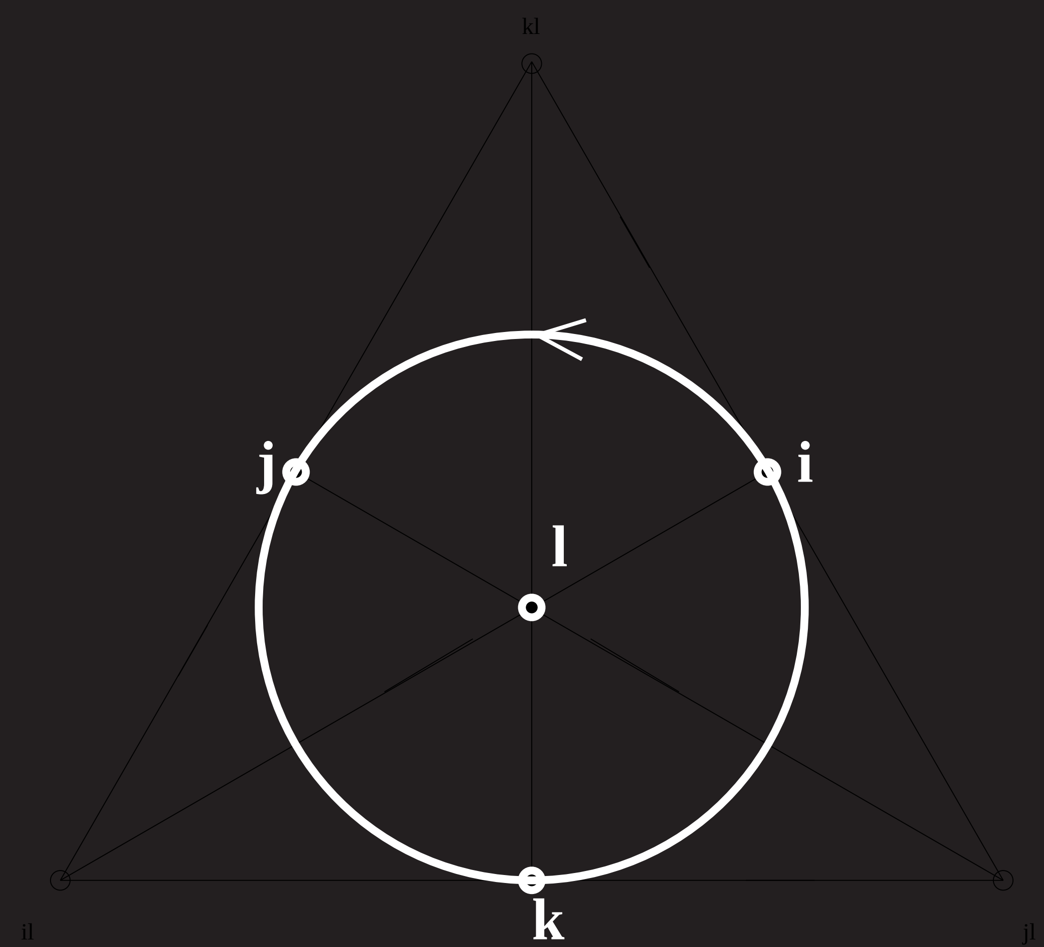
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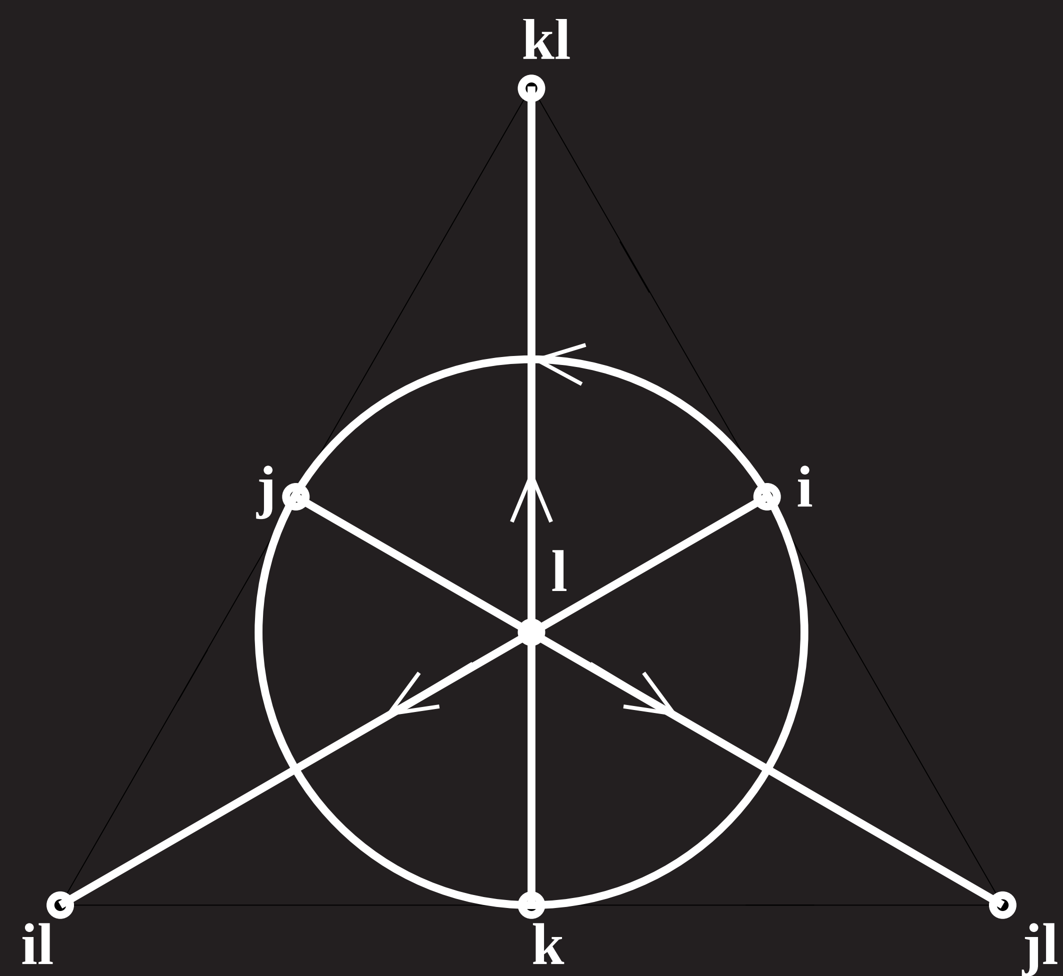
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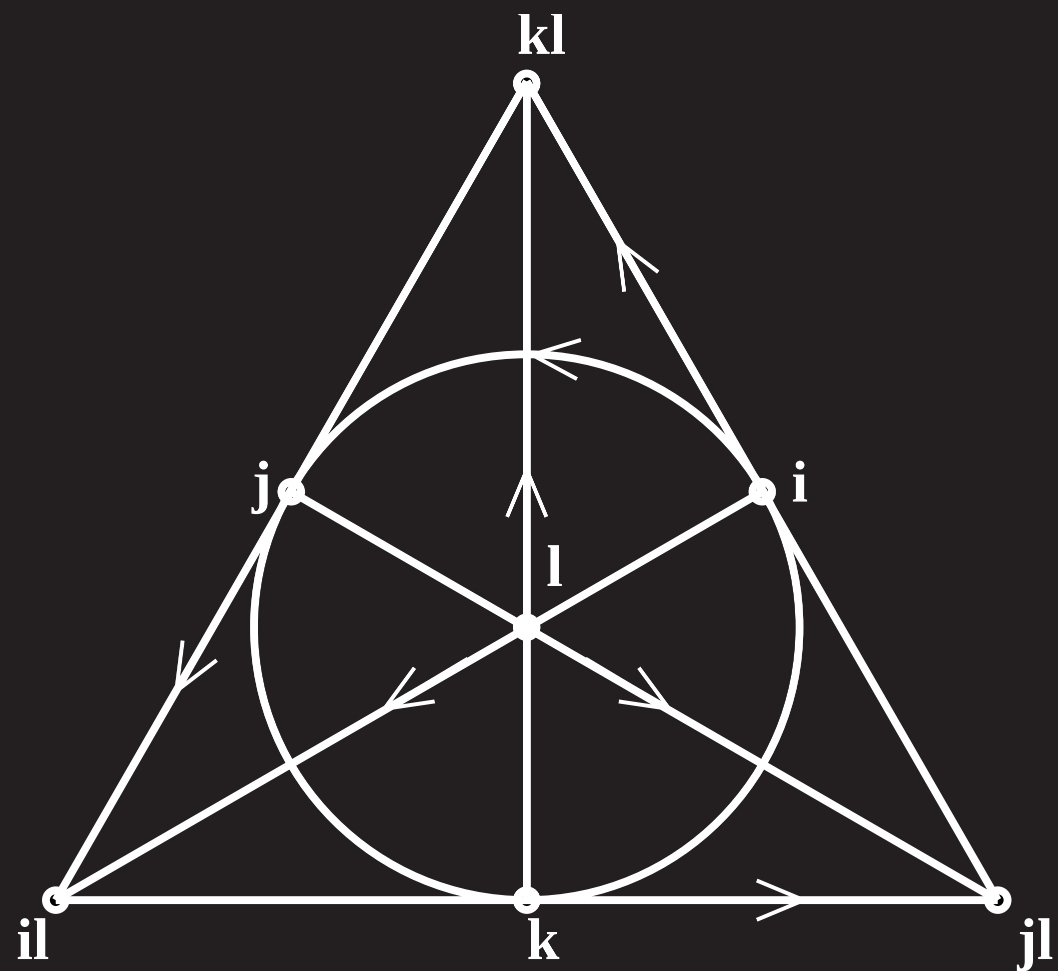
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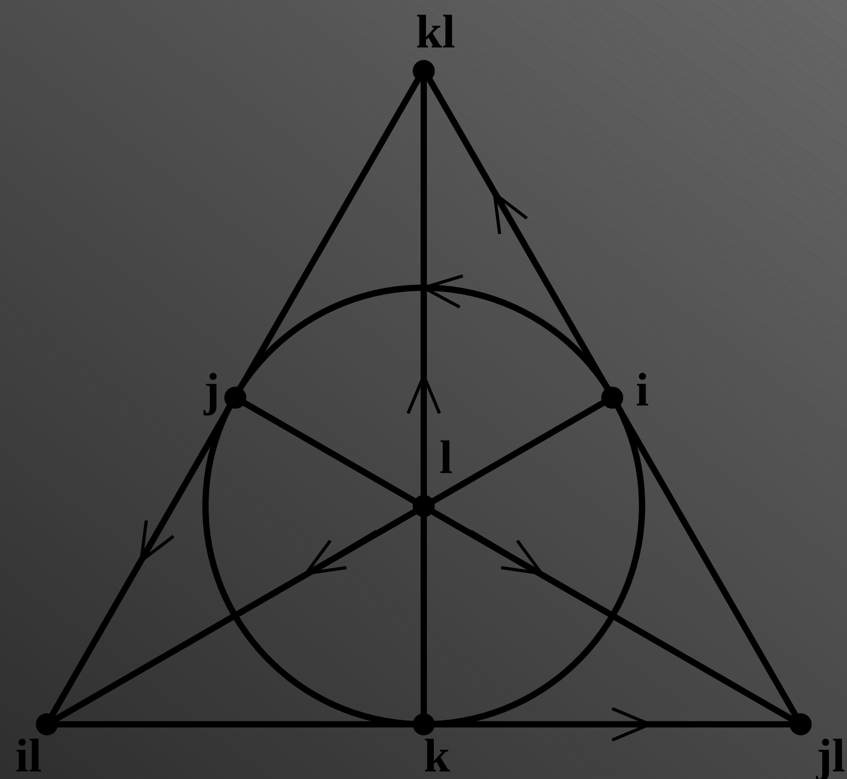
OCTONIONS

each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative



WHAT STILL WORKS?

- $q \in \operatorname{Im} \mathbb{O} \Longrightarrow \overline{q} = -q$
- $q\overline{q} = |q|^2$
- $q^{-1} = \frac{\overline{q}}{|q|^2} \qquad (q \neq 0)$
- $|pq| = |p||q|$

required for supersymmetry

- $[p, p, q] = (pp)q - p(pq) = 0$

alternativity

EXPONENTIAL FORM

$$p = |p|\, e^{s\,\phi} \qquad (s^2 = -1)$$

$$= |p|\,(\cos\phi + s\sin\phi)$$

$$e^{k\phi} i = i e^{-k\phi}$$

$$e^{k\phi} i e^{-k\phi} = i e^{-2k\phi}$$

ROTATIONS

$$x \in \mathbb{H} \quad \longleftrightarrow \quad \vec{x} \in \mathbb{R}^4$$

$$|x| \quad \longleftrightarrow \quad |\vec{x}|$$

$$|p| = 1 \Longrightarrow |px| = |x|$$

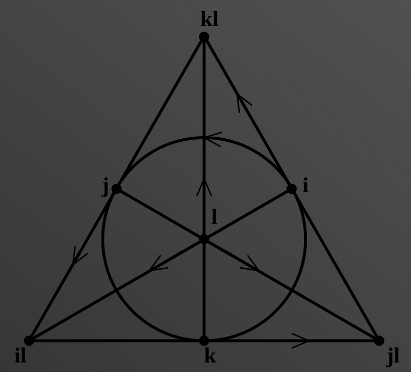
***SO*(3)**:

θ about k-axis: $e^{k\theta}x$

***SO*(7)**?

$e^{k\theta}$ rotates 3 planes perpendicular to k ...

Need ***Flips***



ROTATIONS

$$x \in \mathbb{O} \quad \longleftrightarrow \quad \vec{x} \in \mathbb{R}^8$$

$$|x| \quad \longleftrightarrow \quad |\vec{x}|$$

$$|p| = 1 \Longrightarrow |px\bar{p}| = |x|$$

***SO*(7)**:

2θ about "k-axis": $e^{k\theta} x e^{-k\theta}$

flips (2θ in ij-plane):

$$(i\cos\theta + j\sin\theta)(ixi)(i\cos\theta + j\sin\theta)$$

nesting!

ROTATIONS

$$x \in \mathbb{O} \quad \longleftrightarrow \quad \vec{\boldsymbol{x}} \in \mathbb{R}^8$$

$$|x| \quad \longleftrightarrow \quad |\vec{\boldsymbol{x}}|$$

$$|p| = 1 \Longrightarrow |pxp| = |x|$$

$\boldsymbol{SO}(\mathbf{8})$:

flips OK

2θ in 1ℓ-plane: $e^{\ell\theta} x e^{\ell\theta}$

triality: pxp, px, xp

fails over $\mathbb{H}$!

AUTOMORPHISMS

$$|pxp^{-1}| = |x|$$

$\boldsymbol{SO(3)}$:

over $\mathbb{H}$: $(pxp^{-1})(pyp^{-1}) = p(xy)p^{-1} \quad (\forall p)$

$\boldsymbol{G_2 \subset SO(7)}$:

$p = e^{\pi s/3} \qquad (s^2 = -1)$

$\dim G_2 = 14$

$SU(3) \subset G_2$

$\pi/3$ phase $\longleftrightarrow$ quarks?

Fix 1;

Rotate i: 6 choices;

Rotate j: 5 choices;

k fixed;

Rotate ℓ: 3 choices;

Done.

VECTORS II

$$\boldsymbol{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$\boldsymbol{X}^\dagger = \overline{\boldsymbol{X}}^T = \boldsymbol{X}$$

$$-\det(\boldsymbol{X}) = -t^2 + x^2 + y^2 + z^2$$

- {vectors in (3+1)-dimensional spacetime} $\longleftrightarrow$ {2×2 complex Hermitian matrices}
- determinant $\longleftrightarrow$ (Lorentzian) inner product
- $\boldsymbol{X} = tI + x\sigma_x + y\sigma_y + z\sigma_z$ (Pauli matrices)

LORENTZ TRANSFORMATIONS

Exploit (local) isomorphism:

$$SO(3,1) \approx SL(2,\mathbb{C})$$

$$\boldsymbol{x}' = \boldsymbol{\Lambda x} \quad \longleftrightarrow \quad \boldsymbol{X}' = \boldsymbol{M X M}^\dagger$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\det(\boldsymbol{M}) = 1 \quad \Longrightarrow \quad \det \boldsymbol{X}' = \det \boldsymbol{X}$$

ROTATIONS

$$\boldsymbol{M}_{zx} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\boldsymbol{M}_{xy} = \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\boldsymbol{M}_{yz} = \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$i \longrightarrow j, k, \ldots, \ell$$

ROTATIONS

Still missing: rotations in $\mathrm{Im}\,\mathbb{K}$

$\mathbb{H}$: $\boldsymbol{M} = e^{k\theta}\boldsymbol{I}$

$\mathbb{O}$: flips!

$$\boldsymbol{X}' = \boldsymbol{M}_2(\boldsymbol{M}_1\boldsymbol{X}\boldsymbol{M}_1^\dagger)\boldsymbol{M}_2^\dagger$$

$$\boldsymbol{M}_1 = i\boldsymbol{I} \qquad \boldsymbol{M}_2 = (i\cos\theta + j\sin\theta)\boldsymbol{I}$$

WHICH DIMENSIONS?

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \longmapsto$

$$\boldsymbol{X} = \begin{pmatrix} p & \bar{a} \\ a & m \end{pmatrix} \qquad (p, m \in \mathbb{R}; a \in \mathbb{K})$$

$\dim \mathbb{K} + 2 = 3, 4, 6, 10$ spacetime dimensions

supersymmetry

$$SO(5,1) \approx SL(2,\mathbb{H})$$
$$SO(9,1) \approx SL(2,\mathbb{O})$$

PENROSE SPINORS

$$v = \begin{pmatrix} c \\ \bar{b} \end{pmatrix}$$

$$vv^\dagger = \begin{pmatrix} |c|^2 & cb \\ \bar{b}\bar{c} & |b|^2 \end{pmatrix}$$

$\det(vv^\dagger) = 0$

$(\text{spinor})^2 = \text{null vector}$

Lorentz transformation:

$$\begin{aligned} v' &= \boldsymbol{M} v \\ \boldsymbol{M}(vv^\dagger)\boldsymbol{M}^\dagger &= (\boldsymbol{M}v)(\boldsymbol{M}v)^\dagger \end{aligned}$$

compatibility

HOPF FIBRATIONS

$$v \in \mathbb{K}^2$$

$$v^\dagger v = 1 \Longrightarrow v \in \mathbb{S}^{2k-1}$$

$$\operatorname{tr}(vv^\dagger) = v^\dagger v = 1 \Longrightarrow t = \text{const}$$

$$\det(vv^\dagger) = 0 \Longrightarrow vv^\dagger \in \mathbb{S}^k$$

$$\begin{array}{c} \mathbb{S}^{2k-1} \\ \Big\downarrow \; \mathbb{S}^{k-1} \\ \mathbb{S}^k \end{array}$$

phase freedom: $\begin{pmatrix} p \\ q \end{pmatrix} \longmapsto \begin{pmatrix} \frac{p\bar{q}}{|q|} \\ |q| \end{pmatrix} e^{s\theta}$ $\qquad (s^2 = -1)$

$$\mathbb{O}\boldsymbol{P}^1 = \{(p,q) \sim (pq^{-1}\chi, \chi)\}$$

$$\boxed{\mathbb{O}\boldsymbol{P}^1 = \{\boldsymbol{X}^2 = \boldsymbol{X}, \operatorname{tr}\boldsymbol{X} = 1\}}$$

WEYL EQUATION

- **Massless, relativistic, spin $\frac{1}{2}$**
- **Momentum space**

$$\widetilde{\boldsymbol{P}}\psi = p^{\mu}\widetilde{\boldsymbol{\sigma}}_{\mu}\,\psi = 0$$

$$\widetilde{\boldsymbol{P}} = \boldsymbol{P} - (\operatorname{tr}\boldsymbol{P})\,\boldsymbol{I}$$

Pauli Matrices:

$$\boldsymbol{\sigma}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_\ell = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$$

SOLUTION

Weyl equation: (3 of 4 string equations!)

$$-\widetilde{P}\psi = 0 \qquad \Longrightarrow \det(P) = 0$$

One solution: (P, θ complex)

$$P = \pm\theta\theta^\dagger$$

$$\widetilde{\theta\theta^\dagger}\theta = (\theta\theta^\dagger - \theta^\dagger\theta)\theta = \theta\theta^\dagger\theta - \theta^\dagger\theta\theta = 0$$

General solution: ($\xi \in \mathbb{O}$)

$$\psi = \theta\xi$$

P, ψ quaternionic

PHASE FREEDOM

$$\boldsymbol{P} = \begin{pmatrix} |c|^2 & cb \\ \bar{b}\bar{c} & |b|^2 \end{pmatrix} = \theta\theta^\dagger = (\theta e^{s\phi})(\theta e^{s\phi})^\dagger$$

$$\theta = \begin{pmatrix} |c| \\ \frac{\bar{b}\bar{c}}{|c|} \end{pmatrix}$$

$$\Longrightarrow \widetilde{\boldsymbol{P}} = \begin{pmatrix} -|b|^2 & cb \\ \bar{b}\bar{c} & -|c|^2 \end{pmatrix} \Longrightarrow \psi = \theta\xi = \begin{pmatrix} |c| \\ \frac{\bar{b}\bar{c}}{|c|} \end{pmatrix} r\, e^{s\alpha}$$

- Phase freedom is supersymmetry.
- Solutions are quaternionic (only 2 directions: $\bar{b}\bar{c}$, s).

DIRAC EQUATION

Gamma matrices: ($\mathbb{C}$ *Weyl* representation)

$$\gamma_\mu = \begin{pmatrix} 0 & \widetilde{\boldsymbol{\sigma}}_\mu \\ \boldsymbol{\sigma}_\mu & 0 \end{pmatrix}$$

Dirac equation: (momentum space)

$$(p^\mu \gamma_\mu - m)\,\Psi = 0$$

Weyl equation: ($\mathbb{H}$ Penrose spinors)

$$\Psi = \begin{pmatrix} \theta \\ \eta \end{pmatrix} \longleftrightarrow \psi = \eta + \boldsymbol{\sigma}_k \theta$$

$$(p^\mu\, \widetilde{\boldsymbol{\sigma}}_\mu - m \widetilde{\boldsymbol{\sigma}}_k)\ \psi = 0$$

COMPARISON

4×4 complex:

$$0 = \left(\gamma_t \gamma_\mu \, p^\mu - m \, \gamma_t\right) \Psi$$

2×2 quaternionic:

$$\begin{aligned} 0 &= \left(p^t \sigma_t - p^\alpha \sigma_\alpha - m \, \sigma_k\right) \psi \\ &= -\widetilde{P} \psi \end{aligned}$$

Isomorphism: $(\mathbb{H}^2 \approx \mathbb{C}^4)$

$$\begin{pmatrix} c - kb \\ d + ka \end{pmatrix} \longleftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

DIMENSIONAL REDUCTION

$$SO(3,1) \approx SL(2,\mathbb{C}) \subset SL(2,\mathbb{O}) \approx SO(9,1)$$

Projection: $(\mathbb{O} \to \mathbb{C})$

$$\pi(p) = \frac{1}{2}(p + \ell p \bar{\ell})$$

Determinant: $\det(P) = 0 \Longrightarrow$

$$\det\left(\pi(P)\right) = m^2$$

Mass Term: $P = \pi(P) + m\,\sigma_k \qquad \sigma_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$

$$P = \begin{pmatrix} p^t + p^z & p^x - \ell p^y - km \\ p^x + \ell p^y + km & p^t - p^z \end{pmatrix}$$

SPIN

Finite rotation:

$$R_z = \begin{pmatrix} e^{\ell \frac{\theta}{2}} & 0 \\ 0 & e^{-\ell \frac{\theta}{2}} \end{pmatrix}$$

Infinitesimal rotation:

$$L_z = \left. \frac{dR_z}{d\theta} \right|_{\theta=0} = \frac{1}{2} \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

Right self-adjoint operator:

$$\hat{L}_z \psi := (L_z \psi)\overline{\ell}$$

Right eigenvalue problem:

$$\hat{L}_z \psi = \psi \lambda$$

ANGULAR MOMENTUM REVISITED

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \qquad L_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$L_z = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix} \qquad \hat{L}_\mu \psi := -(L_\mu \psi)\ell$$

$$\psi = e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow$$

$$\hat{L}_z \psi = \psi \frac{1}{2} \qquad \hat{L}_x \psi = -\psi \frac{k}{2} \qquad \hat{L}_y \psi = -\psi \frac{k\ell}{2}$$

Simultaneous eigenvector!

(only 1 *real* eigenvalue)

LEPTONS

ψ $\qquad$ $P = \psi\psi^\dagger$

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad \nu_{-z} \nu_{-z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

How Many Quaternionic Spaces?

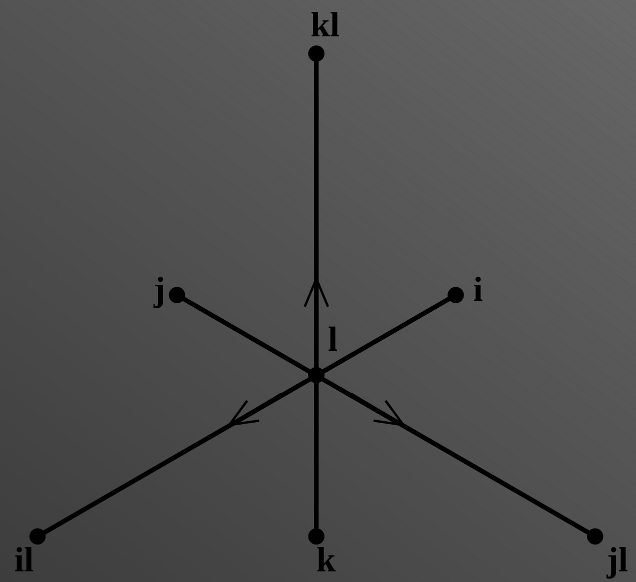
Dimensional Reduction:

$$\Longrightarrow \ell \in \mathbb{H}$$

Orthogonality:

$$(\mathbb{H}_1 \cap \mathbb{H}_2 = \mathbb{C})$$

$$\longmapsto i, j, k$$



Answer: 3!

LEPTONS

$$e_{\uparrow} = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_{\uparrow} e_{\uparrow}^{\dagger} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_{\downarrow} = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_{\downarrow} e_{\downarrow}^{\dagger} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_{z} = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_{z} \nu_{z}^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad \nu_{-z} \nu_{-z}^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\emptyset_{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \emptyset_{z} \emptyset_{z}^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

WHAT NEXT?

Have:

3 generations of leptons!

Neutrinos have just one helicity!

What about $\emptyset_z$?

Want:

- interactions
- quarks/color ($SU(3)$!)
- charge

JORDAN ALGEBRAS

$$\mathcal{X} = \begin{pmatrix} p & a & \overline{c} \\ \overline{a} & m & b \\ c & \overline{b} & n \end{pmatrix}$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\mathcal{I} \end{aligned}$$

JORDAN ALGEBRAS

$u, v, w \in \mathbb{R}^3 \Longrightarrow$

$$
\begin{aligned}
2\, uu^\dagger \circ vv^\dagger &= (u \cdot v)\, (uv^\dagger + vu^\dagger) \\
\mathrm{tr}\,(uu^\dagger \circ vv^\dagger) &= (u \cdot v)^2 \\
2\, uu^\dagger * vv^\dagger &= (u \times v)(u \times v)^\dagger
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}\,(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\
\mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\
&\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I}
\end{aligned}
$$

JORDAN ALGEBRAS

Exceptional quantum mechanics:

(Jordan, von Neumann, Wigner)

$$(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{X}^2 = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{X}^2)$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I} \end{aligned}$$

MORE ROTATION GROUPS

$$\begin{aligned} \sqrt{3}\,t &= p + m + n \\ 2\sqrt{3}\,w &= p + m - 2n \\ 2z &= p - m \end{aligned}$$

$\boldsymbol{SO(27)}$:

$$\mathrm{tr}\,(\mathcal{X} \circ \mathcal{X}) = 2(|a|^2 + |b|^2 + |c|^2 + w^2 + z^2) + t^2$$

$\boldsymbol{SO(26,1)}$:

$$-\mathrm{tr}\,(\mathcal{X} * \mathcal{X}) = |a|^2 + |b|^2 + |c|^2 + w^2 + z^2 - t^2$$

EXCEPTIONAL GROUPS

$\boldsymbol{F_4}$: "$SU(3,\mathbb{O})$"

$$(\mathcal{M}\mathcal{X}\mathcal{M}^\dagger)\circ(\mathcal{M}\mathcal{Y}\mathcal{M}^\dagger) = \mathcal{M}(\mathcal{X}\circ\mathcal{Y})\mathcal{M}^\dagger$$

$\boldsymbol{E_6}$: "$SL(3,\mathbb{O})$"

$$\det\mathcal{X} = \frac{1}{3}\,\mathrm{tr}\,((\mathcal{X}*\mathcal{X})\circ\mathcal{X})$$

$$SO(3,1)\times U(1)\times SU(2)\times SU(3)\subset E_6$$

COMPLEX EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = \lambda v$

Hermitian: $A^\dagger = \overline{A}^T = A$

Reality: $\overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v$

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

since: $\lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 = v_1^\dagger A v_2 = v_1^\dagger \lambda_2 v_2$

- $\exists\, n$ eigenvalues, all real.
- $\exists$ basis of orthonormal eigenvectors.

Decomposition:

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$$

OCTONIONIC EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = v\lambda$

Hermitian: $A^\dagger = A$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} j \\ \ell \end{pmatrix} = \begin{pmatrix} j + i\ell \\ \ell - k \end{pmatrix} = \begin{pmatrix} j \\ \ell \end{pmatrix} (1 - k\ell)$$

Eigenvalues need not be real!

$$\begin{aligned} \overline{\lambda}(v^\dagger v) \quad &\neq (\overline{\lambda} v^\dagger)v = (Av)^\dagger v = (v^\dagger A)v \\ &\neq v^\dagger (Av) = v^\dagger (v\lambda) \neq (v^\dagger v)\lambda \end{aligned}$$

$3{\times}3$ REAL EIGENVALUE PROBLEM

Characteristic Equation:

$$\mathcal{A}^3 - (\operatorname{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det\mathcal{A})\,I = 0$$

$$\mathcal{A}^3 = \frac{1}{2}(\mathcal{A}\mathcal{A}^2 + \mathcal{A}^2\mathcal{A})$$

But:

$$\lambda^3 - (\operatorname{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - \det\mathcal{A} = r$$
$$(r - r_+)(r - r_-) = 0$$

- 2 sets of 3 real eigenvalues!
- 4-parameter families of eigenvectors
- $\det\mathcal{A} = 0 \quad \not\Longrightarrow \quad \lambda = 0$!

DECOMPOSITIONS I

Family: $K[v] = rv =$

$$\mathcal{A}\Big(\mathcal{A}\big(\mathcal{A}(v)\big)\Big) - \mathrm{tr}\mathcal{A}\,\mathcal{A}\big(\mathcal{A}(v)\big) + \sigma(\mathcal{A})\,\mathcal{A}(v) - \det(\mathcal{A})\,v$$

Theorem: (v, w in same family; $\lambda \neq \mu$)

$$(vv^\dagger)w = 0$$

Proof: 8 hour brute force Mathematica computation!

(Analytic proof by Okubo!)

Corollary:

$$\mathcal{A} = \sum \lambda v v^\dagger$$

PROJECTIONS

Idea: $(uu^\dagger)\,\boldsymbol{z} \longleftrightarrow u\,(u \cdot \boldsymbol{z})$

Theorem: ($\boldsymbol{z}$ *any* vector in same family)

$$(uu^\dagger)\left((uu^\dagger)\boldsymbol{z}\right) = (uu^\dagger)\boldsymbol{z}$$

Corollary:

$$(vv^\dagger)\left((uu^\dagger)\boldsymbol{z}\right) = 0$$

Corollary:

$$\mathcal{A}\left((uu^\dagger)\boldsymbol{z}\right) = \lambda\left((uu^\dagger)\boldsymbol{z}\right)$$

DECOMPOSITIONS II

Into Families:

$$\boldsymbol{z} = \frac{K[\boldsymbol{z}] - r_2\,\boldsymbol{z}}{r_1 - r_2} - \frac{K[\boldsymbol{z}] - r_1\,\boldsymbol{z}}{r_1 - r_2} = \boldsymbol{z}_1 + \boldsymbol{z}_2$$

Within Families: $(uu^\dagger + vv^\dagger + ww^\dagger = I)$

$$\boldsymbol{z}_1 = (uu^\dagger)\boldsymbol{z}_1 + (vv^\dagger)\boldsymbol{z}_1 + (ww^\dagger)\boldsymbol{z}_1$$

Theorem:

$$\boldsymbol{z} = \sum (uu^\dagger)\boldsymbol{z}_1 + \sum (\hat{u}\hat{u}^\dagger)\boldsymbol{z}_2$$

This decomposes *any* vector $\boldsymbol{z}$ into *six* eigenvectors, one for each eigenvalue of $\mathcal{A}$!

JORDAN EIGENVALUE PROBLEM

Jordan product:

$$\mathcal{A} \circ \mathcal{B} = \frac{1}{2} \left(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} \right)$$

(**over** $\mathbb{R}$**:** $2\, uu^\dagger \circ vv^\dagger = (u \cdot v)\,(uv^\dagger + vu^\dagger)$)

Eigenvalue problem: (eigenmatrices!)

$$\mathcal{A} \circ \boldsymbol{x} = \lambda \boldsymbol{x}$$

- eigenvalues satisfy characteristic equation
  ($\mathcal{A}^3 \equiv \mathcal{A} \circ \mathcal{A} \circ \mathcal{A}$!)
- "only" 3 eigenvalues
- $\lambda \neq \mu \Longrightarrow \mathcal{V} \circ \mathcal{W} = 0$
- $\mathcal{A} = \sum \lambda \mathcal{V}$

OCTONIONIC EIGENVALUE PROBLEM

- Eigenvalues not necessarily real!
- New notion of orthogonality:

$$(vv^\dagger)w = 0$$

- 6 eigenvalues in 3×3 case!
- Decomposition:

$$\mathcal{A} = \sum \lambda v v^\dagger$$

SUPERSPINORS

$$\mathcal{X} = \begin{pmatrix} \boldsymbol{X} & \theta \\ \theta^\dagger & n \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{M}\mathcal{X}\mathcal{M}^\dagger = \begin{pmatrix} \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^\dagger & \boldsymbol{M}\theta \\ \theta^\dagger \boldsymbol{M}^\dagger & n \end{pmatrix}$$

Cayley/Moufang plane:

$$\begin{aligned} \mathbb{O}\boldsymbol{P}^2 &= \{\mathcal{X}^2 = \mathcal{X}, \operatorname{tr}\mathcal{X} = 1\} \\ &= \{\mathcal{X} * \mathcal{X} = 0, \operatorname{tr}\mathcal{X} = 1\} \\ &= \{\mathcal{X} = \psi\psi^\dagger, \psi^\dagger\psi = 1\} \end{aligned}$$

$$(\psi \in \mathbb{H}^3)$$

DIRAC EQUATION II

$$\mathcal{P} = \begin{pmatrix} \boldsymbol{P} & \theta\xi \\ \bar{\xi}\theta^\dagger & |\xi|^2 \end{pmatrix}$$

$$\mathcal{P} * \mathcal{P} = 0 \Longrightarrow \widetilde{\boldsymbol{P}}\theta = 0$$

$$\mathcal{P} = \psi\psi^\dagger$$

quaternionic!

Furthermore: $$\mathcal{X}^\dagger = \mathcal{X} \Longrightarrow \mathcal{X} = \sum_{n=1}^{3} \psi_n \psi_n^\dagger$$

leptons, mesons, baryons?

THE END