The GEOMETRY of the OCTONIONS

OSU

Oregon State
UNIVERSITY

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DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$
$$z = x + yi$$

$$\boxed{i^2 = -1}$$

DIVISION ALGEBRAS

Real Numbers:

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Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$
$$q = (a + bi) + (c + di)j$$

Complex Numbers:

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$$\boxed{i^2 = j^2 = -1}$$

● i

j● ●i
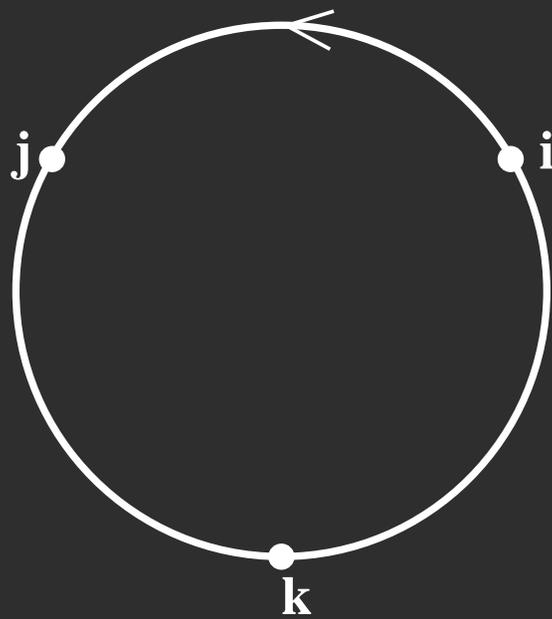
QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

not commutative
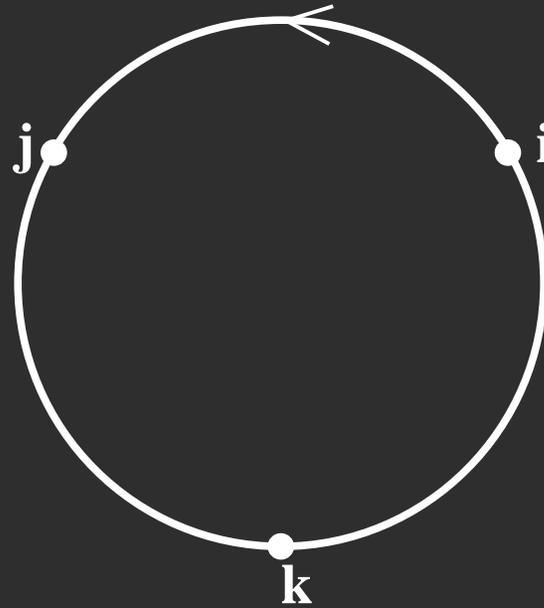
THE DISCOVERY OF THE QUATERNIONS
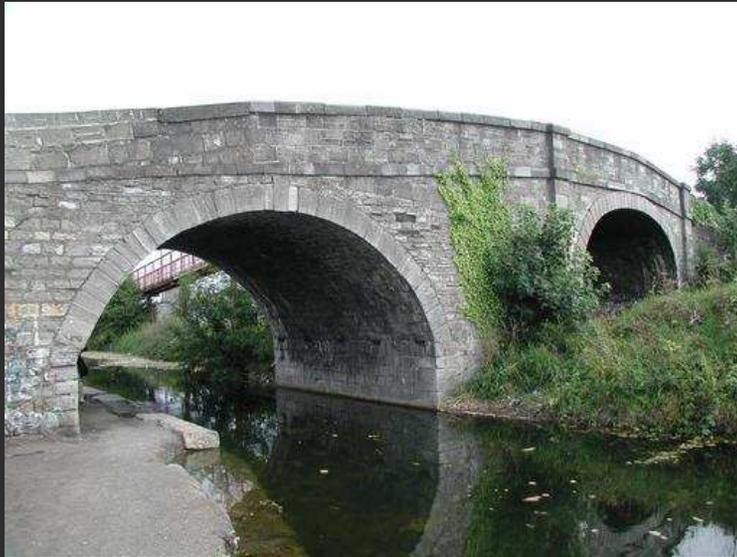
Brougham Bridge (Dublin)

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$$i^2 = j^2 = k^2 = ijk = -1$$
& cut it on a stone of this bridge

VECTORS I

$$v = bi + cj + dk \longleftrightarrow \vec{v} = b\,\hat{\boldsymbol{\imath}} + c\,\hat{\boldsymbol{\jmath}} + d\,\hat{\boldsymbol{k}}$$

$$vw \longleftrightarrow -\vec{v}\cdot\vec{w} + \vec{v}\times\vec{w}$$

Dot product exists in any dimension
but
Cross product exists only in 3 and 7 dimensions

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Octonions:

$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

$$i^2 = j^2 = \ell^2 = -1$$
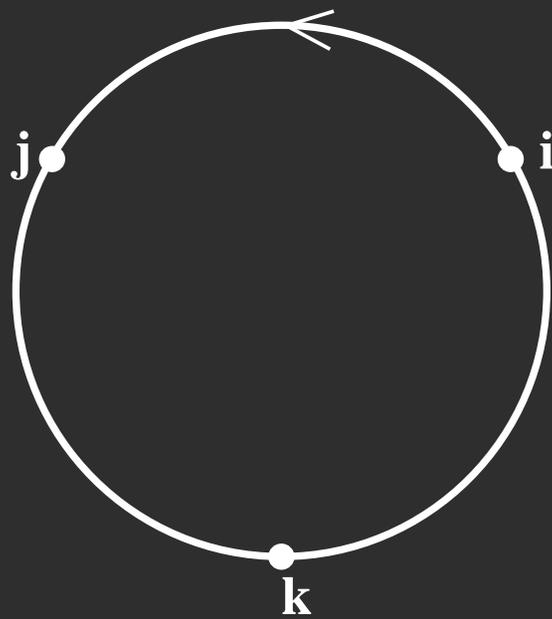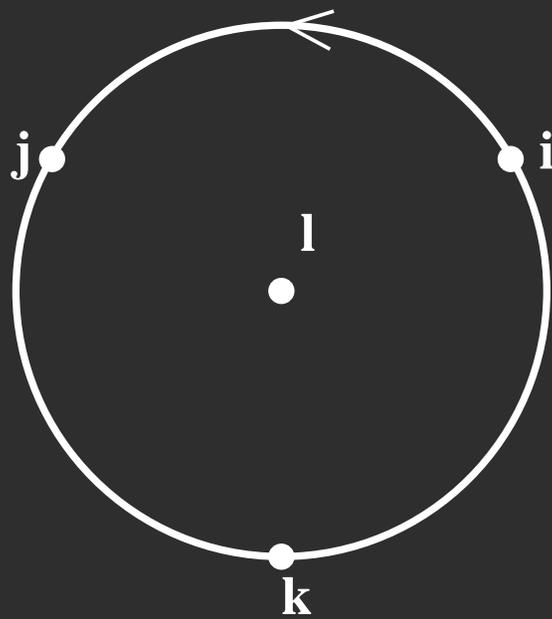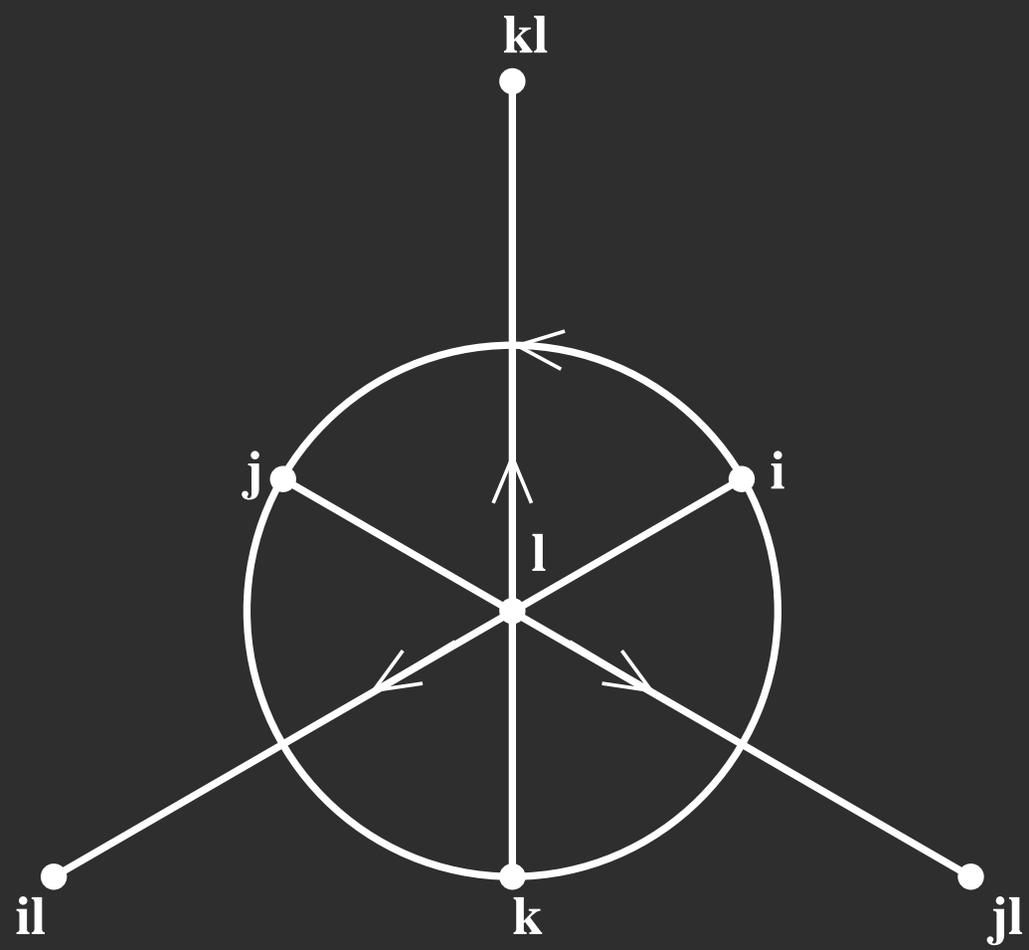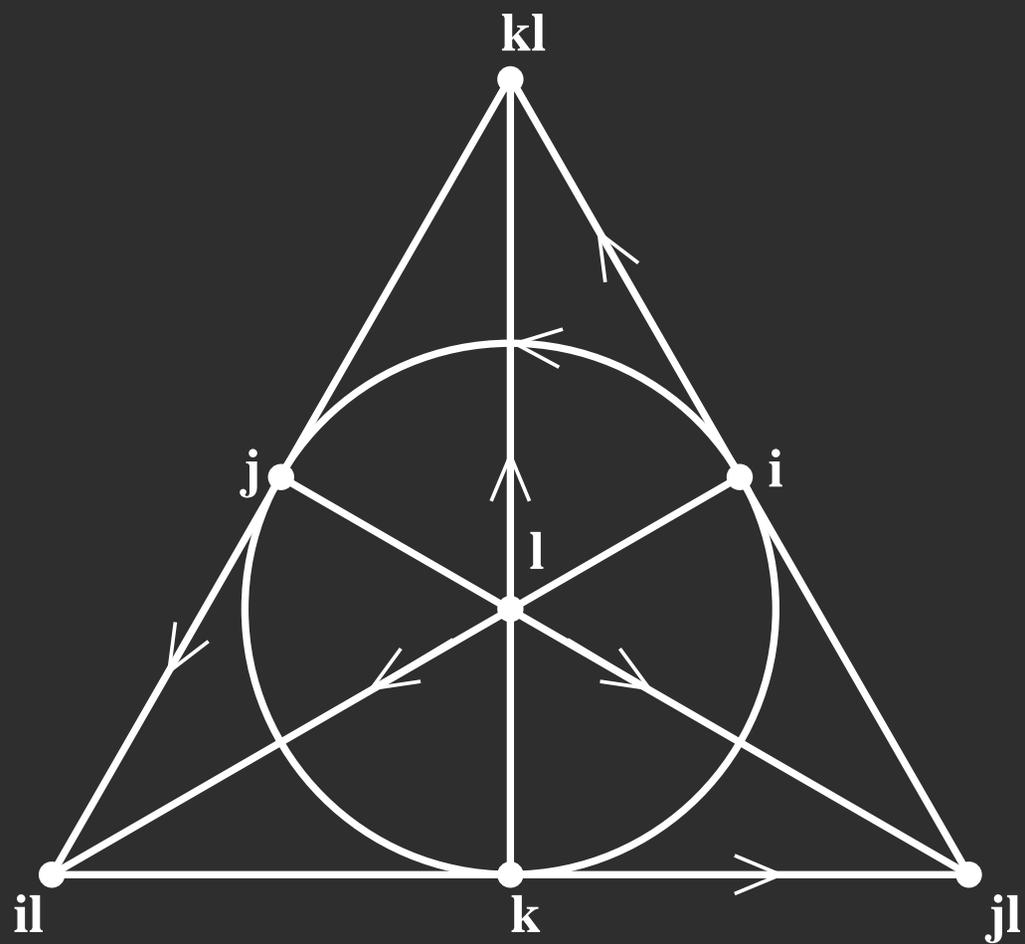
OCTONIONS

each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative
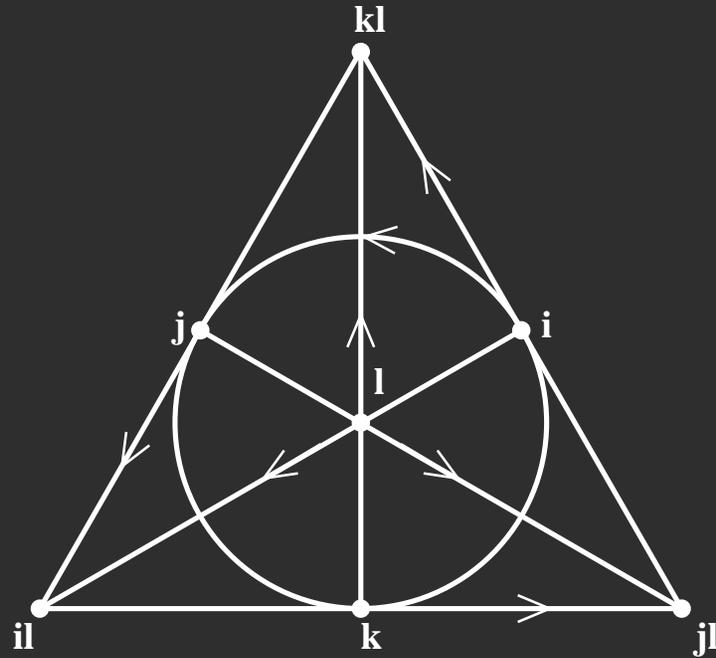
THE DISCOVERY OF THE OCTONIONS

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John T. Graves (1843!)
Arthur Cayley (1845)
octaves, Cayley numbers

WHAT STILL WORKS?

- $q \in \operatorname{Im} \mathbb{O} \implies \bar{q} = -q$

- $q\bar{q} = |q|^2$

- $q^{-1} = \dfrac{\bar{q}}{|q|^2} \qquad (q \neq 0)$

- $|pq| = |p||q|$

  required for supersymmetry

- $[p, p, q] = (pp)q - p(pq) = 0$

  alternativity

EXPONENTIAL FORM

$$p = |p|\, e^{s\,\phi} \qquad (s^2 = -1)$$

$$= |p|\, (\cos\phi + s\sin\phi)$$

$$e^{k\phi}i = i e^{-k\phi}$$

$$e^{k\phi}i e^{-k\phi} = i e^{-2k\phi}$$

$$\boxed{\textbf{ROTATIONS}}$$

$$x \in \mathbb{H} \longleftrightarrow \vec{x} \in \mathbb{R}^4$$

$$|x| \longleftrightarrow |\vec{x}|$$

$$|p| = 1 \implies |px| = |x|$$

$SO(3)$:

θ about k-axis: $e^{k\theta}x$

$SO(7)$?

$e^{k\theta}$ rotates 3 planes perpendicular to k ...

Need *Flips*
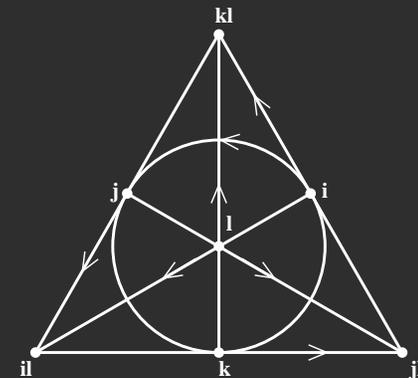
ROTATIONS

$$x \in \mathbb{O} \longleftrightarrow \vec{x} \in \mathbb{R}^8$$

$$|x| \longleftrightarrow |\vec{x}|$$

$$|p| = 1 \implies |px\overline{p}| = |x|$$

$SO(7)$:

2θ about "k-axis": $e^{k\theta} x e^{-k\theta}$

flips (2θ in ij-plane):

$$(i\cos\theta + j\sin\theta)(ixi)(i\cos\theta + j\sin\theta)$$

nesting!

ROTATIONS

$$x \in \mathbb{O} \longleftrightarrow \vec{x} \in \mathbb{R}^8$$

$$|x| \longleftrightarrow |\vec{x}|$$

$$|p| = 1 \Longrightarrow |pxp| = |x|$$

$SO(8)$:

    flips OK

    2θ in 1ℓ-plane: $e^{\ell\theta} x e^{\ell\theta}$

triality: pxp, px, xp fails over $\mathbb{H}$!

AUTOMORPHISMS

$$|pxp^{-1}| = |x|$$

$SO(3)$:
over $\mathbb{H}$: $(pxp^{-1})(pyp^{-1}) = p(xy)p^{-1} \quad (\forall p)$

$G_2 \subset SO(7)$:
$p = e^{\pi s/3} \quad (s^2 = -1)$

$$\boxed{\dim G_2 = 14}$$

$$\boxed{\begin{array}{c} SU(3) \subset G_2 \\ \pi/3 \text{ phase} \longleftrightarrow \text{quarks?} \end{array}}$$

Fix 1;
Rotate i: 6 choices;
Rotate j: 5 choices;
k fixed;
Rotate ℓ: 3 choices;
Done.

VECTORS II

$$\boldsymbol{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \longleftrightarrow \boldsymbol{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$\boldsymbol{X}^{\dagger} = \overline{\boldsymbol{X}}^{T} = \boldsymbol{X}$$

$$-\det(\boldsymbol{X}) = -t^2 + x^2 + y^2 + z^2$$

- {vectors in (3+1)-dimensional spacetime}
  $\longleftrightarrow$ {2×2 complex Hermitian matrices}

- determinant $\longleftrightarrow$ (Lorentzian) inner product

- $\boldsymbol{X} = tI + x\sigma_x + y\sigma_y + z\sigma_z$ (Pauli matrices)

LORENTZ TRANSFORMATIONS

Exploit (local) isomorphism:
$$SO(3,1) \approx SL(2,\mathbb{C})$$

$$\boldsymbol{x}' = \boldsymbol{\Lambda}\boldsymbol{x} \qquad \longleftrightarrow \qquad \boldsymbol{X}' = \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^{\dagger}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\det(\boldsymbol{M}) = 1 \qquad \Longrightarrow \qquad \det\boldsymbol{X}' = \det\boldsymbol{X}$$

$$\boxed{\textbf{ROTATIONS}}$$

$$\boldsymbol{M}_{zx} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\boldsymbol{M}_{xy} = \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\boldsymbol{M}_{yz} = \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\boxed{i \longrightarrow j, k, ..., \ell}$$

$$\boxed{\textbf{ROTATIONS}}$$

Still missing: rotations in $\operatorname{Im}\mathbb{K}$

$\mathbb{H}$: $\boldsymbol{M} = e^{k\theta}\boldsymbol{I}$

$\mathbb{O}$: flips!

$$\boldsymbol{X}' = \boldsymbol{M}_2(\boldsymbol{M}_1\boldsymbol{X}\boldsymbol{M}_1^\dagger)\boldsymbol{M}_2^\dagger$$

$$\boldsymbol{M}_1 = i\boldsymbol{I} \qquad \boldsymbol{M}_2 = (i\cos\theta + j\sin\theta)\boldsymbol{I}$$

WHICH DIMENSIONS?

$$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \longmapsto$$

$$X = \begin{pmatrix} p & \overline{a} \\ a & m \end{pmatrix} \qquad (p, m \in \mathbb{R}; a \in \mathbb{K})$$

$$\dim \mathbb{K} + 2 = 3, 4, 6, 10 \text{ spacetime dimensions}$$

supersymmetry

$$SO(5,1) \approx SL(2, \mathbb{H})$$
$$SO(9,1) \approx SL(2, \mathbb{O})$$

PENROSE SPINORS

$$v = \begin{pmatrix} c \\ b \end{pmatrix}$$

$$\boxed{\det(vv^\dagger) = 0}$$

$$vv^\dagger = \begin{pmatrix} |c|^2 & cb \\ \overline{b}\overline{c} & |b|^2 \end{pmatrix}$$

$$\boxed{(\text{spinor})^2 = \text{null vector}}$$

Lorentz transformation:

$$v' = \boldsymbol{M}v$$

$$\boldsymbol{M}(vv^\dagger)\boldsymbol{M}^\dagger = (\boldsymbol{M}v)(\boldsymbol{M}v)^\dagger$$

$$\boxed{\text{compatibility}}$$

HOPF FIBRATIONS

$$v \in \mathbb{K}^2$$

$$v^\dagger v = 1 \implies v \in \mathbb{S}^{2k-1}$$

$$\operatorname{tr}(vv^\dagger) = v^\dagger v = 1 \implies t = \text{const}$$

$$\det(vv^\dagger) = 0 \implies vv^\dagger \in \mathbb{S}^k$$

$$\mathbb{S}^{2k-1}$$

$$\downarrow \mathbb{S}^{k-1}$$

$$\mathbb{S}^k$$
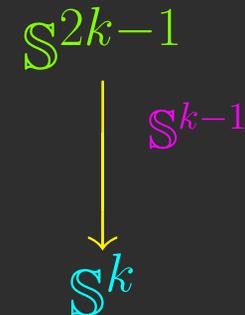
phase freedom:
$$\begin{pmatrix} p \\ q \end{pmatrix} \longmapsto \begin{pmatrix} \frac{p\bar{q}}{|q|} \\ |q| \end{pmatrix} e^{s\theta} \qquad (s^2 = -1)$$

$$\mathbb{O}\boldsymbol{P}^1 = \{(p,q) \sim (pq^{-1}\chi, \chi)\}$$

$$\boxed{\mathbb{O}\boldsymbol{P}^1 = \{\boldsymbol{X}^2 = \boldsymbol{X}, \operatorname{tr}\boldsymbol{X} = 1\}}$$

$$\boxed{\textbf{WEYL EQUATION}}$$

- **Massless, relativistic, spin $\frac{1}{2}$**
- **Momentum space**

$$\widetilde{\boldsymbol{P}}\psi = p^{\mu}\widetilde{\boldsymbol{\sigma}}_{\mu}\,\psi = 0$$

$$\widetilde{\boldsymbol{P}} = \boldsymbol{P} - (\operatorname{tr}\boldsymbol{P})\,\boldsymbol{I}$$

Pauli Matrices:

$$\boldsymbol{\sigma}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_\ell = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$$

SOLUTION

Weyl equation: (3 of 4 string equations!)

$$-\widetilde{P}\psi = 0 \qquad\qquad \Longrightarrow \det(P) = 0$$

One solution: (P, θ complex)

$$P = \pm\theta\theta^{\dagger}$$

$$\widetilde{\theta\theta^{\dagger}}\theta = (\theta\theta^{\dagger} - \theta^{\dagger}\theta)\theta = \theta\theta^{\dagger}\theta - \theta^{\dagger}\theta\theta = 0$$

General solution: ($\xi \in \mathbb{O}$)

$$\psi = \theta\xi$$

P, ψ quaternionic

PHASE FREEDOM

$$\boldsymbol{P} = \begin{pmatrix} |c|^2 & cb \\ \overline{b}\overline{c} & |b|^2 \end{pmatrix} = \theta\theta^\dagger = (\theta e^{s\phi})(\theta e^{s\phi})^\dagger$$

$$\theta = \begin{pmatrix} |c| \\ \frac{\overline{b}\overline{c}}{|c|} \end{pmatrix}$$

$$\Longrightarrow \widetilde{\boldsymbol{P}} = \begin{pmatrix} -|b|^2 & cb \\ \overline{b}\overline{c} & -|c|^2 \end{pmatrix} \Longrightarrow \psi = \theta\xi = \begin{pmatrix} |c| \\ \frac{\overline{b}\overline{c}}{|c|} \end{pmatrix} r\, e^{s\alpha}$$

- Phase freedom is supersymmetry.
- Solutions are quaternionic (only 2 directions: $\overline{b}\overline{c}, s$).

DIRAC EQUATION

Gamma matrices: ($\mathbb{C}$ *Weyl* representation)

$$\gamma_\mu = \begin{pmatrix} 0 & \widetilde{\boldsymbol{\sigma}}_\mu \\ \boldsymbol{\sigma}_\mu & 0 \end{pmatrix}$$

Dirac equation: (momentum space)

$$\left(p^\mu \, \gamma_\mu - m \right) \Psi = 0$$

Weyl equation: ($\mathbb{H}$ Penrose spinors)

$$\Psi = \begin{pmatrix} \theta \\ \eta \end{pmatrix} \longleftrightarrow \psi = \eta + \boldsymbol{\sigma}_k \theta$$

$$\left(p^\mu \, \widetilde{\boldsymbol{\sigma}}_\mu - m \widetilde{\boldsymbol{\sigma}}_k \right) \psi = 0$$

$$\boxed{\textbf{COMPARISON}}$$

4×4 complex:

$$0 = \left(\gamma_t \gamma_\mu \, p^\mu - m \, \gamma_t \right) \Psi$$

2×2 quaternionic:

$$0 = \left(p^t \sigma_t - p^\alpha \sigma_\alpha - m \, \sigma_k \right) \psi$$
$$= -\widetilde{P}\psi$$

Isomorphism: $\left(\mathbb{H}^2 \approx \mathbb{C}^4 \right)$

$$\begin{pmatrix} c - kb \\ d + ka \end{pmatrix} \longleftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

DIMENSIONAL REDUCTION

$$SO(3,1) \approx SL(2,\mathbb{C}) \subset SL(2,\mathbb{O}) \approx SO(9,1)$$

Projection: $(\mathbb{O} \to \mathbb{C})$

$$\pi(p) = \frac{1}{2}(p + \ell p \overline{\ell})$$

Determinant: $\det(P) = 0 \implies$

$$\det\big(\pi(P)\big) = m^2$$

Mass Term: $\qquad P = \pi(P) + m\,\sigma_k \qquad \sigma_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$

$$P = \begin{pmatrix} p^t + p^z & p^x - \ell p^y - km \\ p^x + \ell p^y + km & p^t - p^z \end{pmatrix}$$

$$\boxed{\textbf{SPIN}}$$

Finite rotation:

$$R_z = \begin{pmatrix} e^{\ell\frac{\theta}{2}} & 0 \\ 0 & e^{-\ell\frac{\theta}{2}} \end{pmatrix}$$

Infinitesimal rotation:

$$L_z = \left.\frac{dR_z}{d\theta}\right|_{\theta=0} = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

Right self-adjoint operator:

$$\hat{L}_z\psi := (L_z\psi)\overline{\ell}$$

Right eigenvalue problem:

$$\hat{L}_z\psi = \psi\lambda$$

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \qquad L_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$L_z = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix} \qquad \hat{L}_\mu \psi := -(L_\mu \psi)\ell$$

$$\psi = e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow$$

$$\hat{L}_z \psi = \psi\,\frac{1}{2} \qquad \hat{L}_x \psi = -\psi\,\frac{k}{2} \qquad \hat{L}_y \psi = -\psi\,\frac{k\ell}{2}$$

Simultaneous eigenvector!

(only 1 *real* eigenvalue)

$$\boxed{\textbf{LEPTONS}}$$

$$\boxed{\psi} \qquad \boxed{P = \psi\psi^\dagger}$$

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad\qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad\qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad\qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad\qquad \nu_{-z} \nu_{-z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

How Many Quaternionic Spaces?

Dimensional Reduction:

$$\implies \ell \in \mathbb{H}$$

Orthogonality:

$$(\mathbb{H}_1 \cap \mathbb{H}_2 = \mathbb{C})$$

$$\longmapsto i, j, k$$
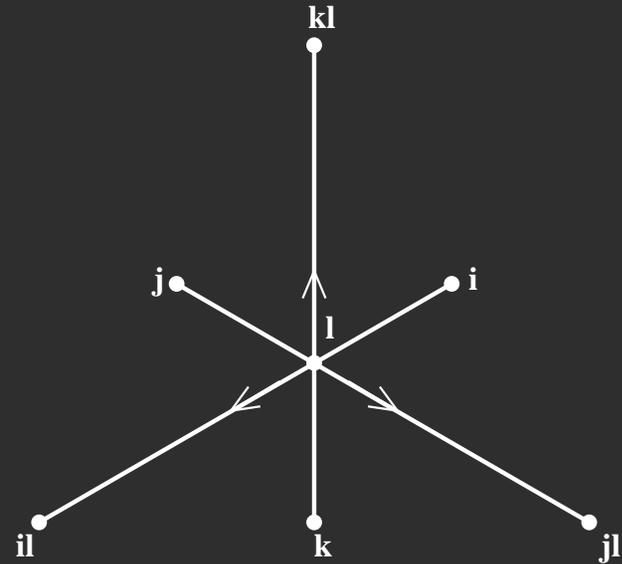
Answer: $3!$

LEPTONS

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad \nu_{-z} \nu_{-z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\varnothing_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \varnothing_z \varnothing_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

WHAT NEXT?

Have:

3 generations of leptons!

Neutrinos have just one helicity!

What about $\varnothing_z$?

Want:

- interactions
- quarks/color ($SU(3)$!)
- charge

JORDAN ALGEBRAS

$$\mathcal{X} = \begin{pmatrix} p & a & \overline{c} \\ \overline{a} & m & b \\ c & \overline{b} & n \end{pmatrix}$$

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}\left(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}\right)$$

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\operatorname{tr}(\mathcal{Y}) + \mathcal{Y}\operatorname{tr}(\mathcal{X})\Big)$$

$$+ \frac{1}{2}\Big(\operatorname{tr}(\mathcal{X})\operatorname{tr}(\mathcal{Y}) - \operatorname{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\mathcal{I}$$

JORDAN ALGEBRAS

$$u, v, w \in \mathbb{R}^3 \implies$$

$$2\, uu^\dagger \circ vv^\dagger = (u \cdot v)\left(uv^\dagger + vu^\dagger\right)$$

$$\mathrm{tr}\left(uu^\dagger \circ vv^\dagger\right) = (u \cdot v)^2$$

$$2\, uu^\dagger * vv^\dagger = (u \times v)(u \times v)^\dagger$$

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X})$$

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big)$$

$$+ \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\mathcal{I}$$

JORDAN ALGEBRAS

Exceptional quantum mechanics:

(Jordan, von Neumann, Wigner)

$$(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{X}^2 = \mathcal{X} \circ \left(\mathcal{Y} \circ \mathcal{X}^2\right)$$

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}\left(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}\right)$$

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\left(\mathcal{Y}\right) + \mathcal{Y}\,\mathrm{tr}\left(\mathcal{X}\right)\Big)$$

$$+ \frac{1}{2}\Big(\mathrm{tr}\left(\mathcal{X}\right)\mathrm{tr}\left(\mathcal{Y}\right) - \mathrm{tr}\left(\mathcal{X} \circ \mathcal{Y}\right)\Big)\mathcal{I}$$

MORE ROTATION GROUPS

$$\sqrt{3}\,t = p + m + n$$
$$2\sqrt{3}\,w = p + m - 2n$$
$$2z = p - m$$

$SO(27)$:
$$\text{tr}\,(\mathcal{X} \circ \mathcal{X}) = 2(|a|^2 + |b|^2 + |c|^2 + w^2 + z^2) + t^2$$

$SO(26,1)$:
$$-\text{tr}\,(\mathcal{X} * \mathcal{X}) = |a|^2 + |b|^2 + |c|^2 + w^2 + z^2 - t^2$$

F_4: "$SU(3, \mathbb{O})$"

$$(\mathcal{M}\mathcal{X}\mathcal{M}^\dagger) \circ (\mathcal{M}\mathcal{Y}\mathcal{M}^\dagger) = \mathcal{M}(\mathcal{X} \circ \mathcal{Y})\mathcal{M}^\dagger$$

E_6: "$SL(3, \mathbb{O})$"

$$\det \mathcal{X} = \tfrac{1}{3} \operatorname{tr}((\mathcal{X} * \mathcal{X}) \circ \mathcal{X})$$

$$SO(3,1) \times U(1) \times SU(2) \times SU(3) \subset E_6$$

SUPERSPINORS

$$\mathcal{X} = \begin{pmatrix} \boldsymbol{X} & \theta \\ \theta^\dagger & n \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{M}\mathcal{X}\mathcal{M}^\dagger = \begin{pmatrix} \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^\dagger & \boldsymbol{M}\theta \\ \theta^\dagger \boldsymbol{M}^\dagger & n \end{pmatrix}$$

Cayley/Moufang plane:

$$\begin{aligned}
\mathbb{O}\boldsymbol{P}^2 &= \{\mathcal{X}^2 = \mathcal{X}, \operatorname{tr}\mathcal{X} = 1\} \\
&= \{\mathcal{X} * \mathcal{X} = 0, \operatorname{tr}\mathcal{X} = 1\} \\
&= \{\mathcal{X} = \psi\psi^\dagger, \psi^\dagger\psi = 1\}
\end{aligned}$$

$$(\psi \in \mathbb{H}^3)$$

DIRAC EQUATION II

$$\mathcal{P} = \begin{pmatrix} \boldsymbol{P} & \theta\xi \\ \overline{\xi}\theta^{\dagger} & |\xi|^2 \end{pmatrix}$$

$$\mathcal{P} * \mathcal{P} = 0 \implies \widetilde{\boldsymbol{P}}\theta = 0$$

$$\mathcal{P} = \psi\psi^{\dagger}$$

quaternionic!

Furthermore:

$$\mathcal{X}^{\dagger} = \mathcal{X} \implies \mathcal{X} = \sum_{n=1}^{3} \psi_n \psi_n^{\dagger}$$

leptons, mesons, baryons?

Life is complex.

It has real and imaginary parts.

Life is octonionic...

THE END