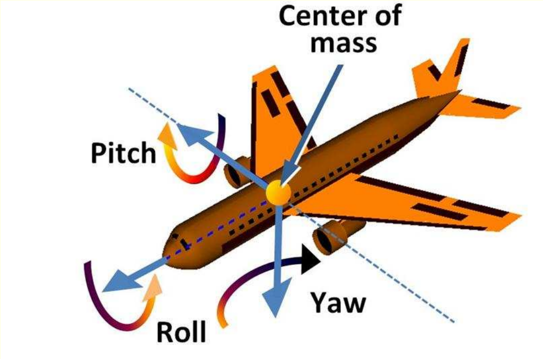
Algebra meets Geometry: Using the Quaternions to Implement Rotations

Tevian Dray

Department of Mathematics
Oregon State University
http://math.oregonstate.edu/~tevian

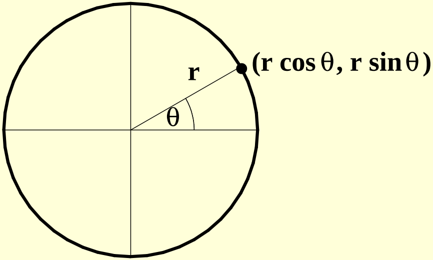
Introduction

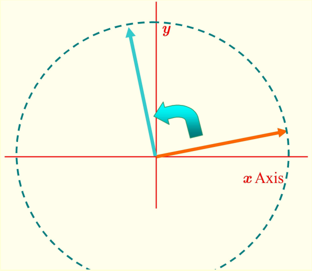


3-d rotations: Aeronautics, robotics, computer graphics, ...

New Content: Use quaternions to implement rotations

Rotations

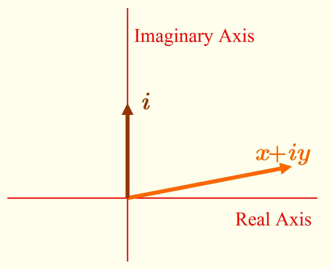




Polar coordinates: $x = r\cos\theta;\ y = r\sin\theta.$

Complex Plane

$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$



$$i^2 = -1$$

$$(x, y) \longmapsto x + iy$$

$$x + iy = r\cos\theta + i\,r\sin\theta = r\,e^{i\theta}$$

Special case: $e^{\pm i\pi/2} = \pm i$

$$e^{i\pi} + 1 = 0$$

Representing Complex Numbers

- Please stand up.
- Use left hand.
- Real axis points forward.
- Imaginary axis points upward.

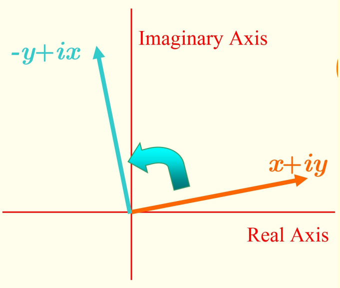
Show me:

- 1
- i
- $2\,i$
- $1 + i$
- $(1 + i)i = i - 1$

- Thank you; please sit down.

Embodied Cognition!

Multiplication by i



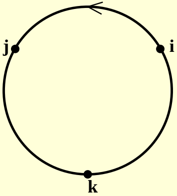
Multiplication by i: $(x + i\,y)i = i\,x + i^2 y = -y + ix$

Multiplication by $e^{i\theta}$: $(r\,e^{i\alpha})e^{i\theta} = r\,e^{i(\alpha+\theta)}$

(Rotates counterclockwise by θ!)

$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$

(1843)



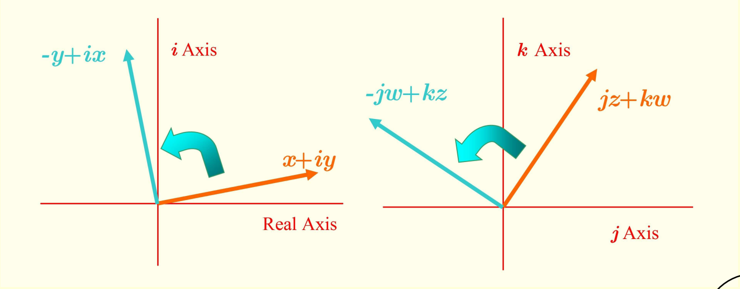
$$q = (x + y\,i) + (z + w\,i)\,j = x + y\,i + z\,j + w\,k$$

$$\boxed{ij = k = -ji;\ i^2 = j^2 = k^2 = -1}$$

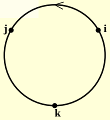
$\mathbb{H}$ is for Hamilton! ($\mathbb{Q}$ denotes rationals)

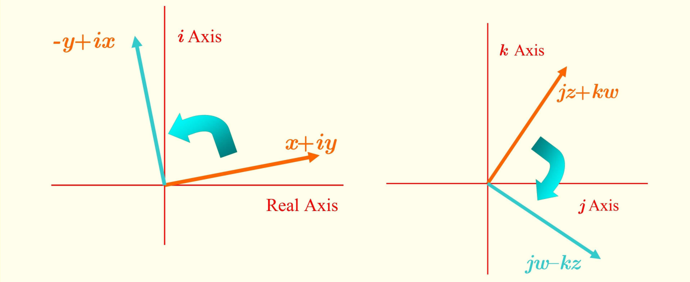
1880s: Vector analysis (Gibbs); $i, j, k \longrightarrow \hat{\imath}, \hat{\jmath}, \hat{k}$

iq vs. qi

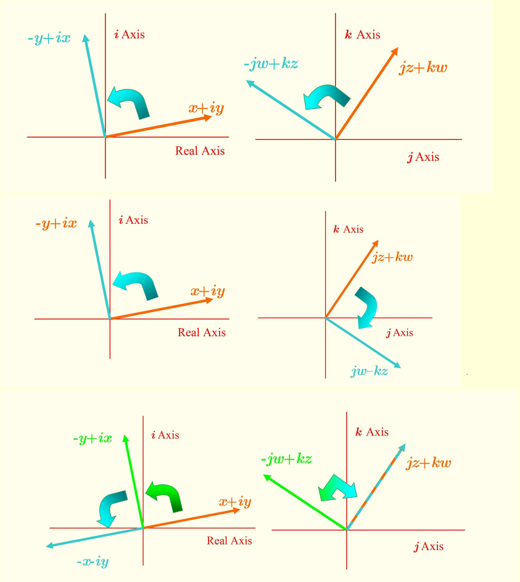


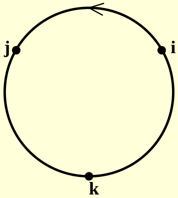
$$q = x + iy + jz + kw$$
$$iq = ix - y + kz - jw$$
$$qi = ix - y - kz + jw$$





Conjugation



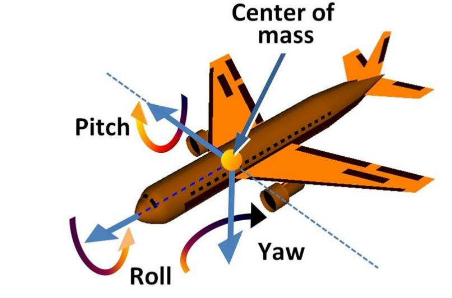


$$q = x + iy + jz + kw$$

$$iq = ix - y + kz - jw$$
$$qi = ix - y - kz + jw$$

$$iqi = -x - iy + jz + kw$$
$$-iqi = x + iy - jz - kw$$
(rotation in jk-plane)

Rotations



$$q \longmapsto e^{i\theta/2} q e^{-i\theta/2}$$

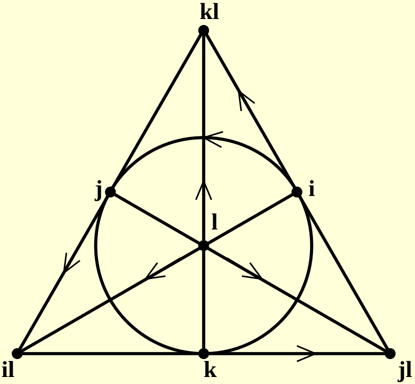
- $1 \longmapsto 1$; $i \longmapsto i$
- Rotates by θ "about i" (in jk-plane)
- $q \longmapsto e^{j\theta/2} q e^{-j\theta/2}$ rotates about j, etc.

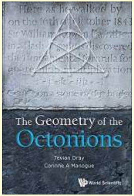
$\therefore$ SO(3), the rigid rotations in 3 dimensions

Generalizations

Plum muffins

Generalizations





2015

Octonions! $(\mathbb{O} = \mathbb{H} + \mathbb{H}\ell)$

Use to model particle physics

http://octonions.geometryof.org/GO