Integrals in Mathematics and Physics

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Acknowledgments

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Mathematics vs. Physics

Two disciplines separated by a common language.

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Mathematicians do algebra; Physicists do geometry.

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Mathematicians do algebra; Physicists do geometry.

$$\vec{\mathbf{F}} = \langle P, Q, R \rangle \qquad \text{vs.} \qquad \vec{\mathbf{F}} = F_x\,\hat{\mathbf{x}} + F_y\,\hat{\mathbf{y}} + F_z\,\hat{\mathbf{z}}$$

$$\vec{\mathbf{G}} = \langle -y, x, 0 \rangle \qquad \text{vs.} \qquad \vec{\mathbf{G}} = -y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} = r\,\hat{\phi}$$

What is an integral?

What is an integral?

- The area under a curve;
- An antiderivative;
- An accumulation.

Integration

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- The area under a curve;
- An antiderivative;
- An accumulation.

What is the integrand?

$$\int x\,dx$$

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$$\int x\,dx \qquad \int \frac{y}{x}\,dA$$

What is an integral?

- The area under a curve;
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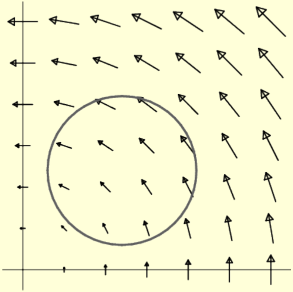
- An accumulation.

What is the integrand?

$$\int x\,dx \qquad \int \frac{y}{x}\,dA \qquad \int\!\!\int \tan\theta\; r\,dr\,d\theta$$

Vector Line Integrals: $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$



$(\vec{\mathbf{F}} = r\,\hat{\phi} = x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}})$

Research Question:

- What does an analysis of textbook treatments of vector line integrals reveal about the learning objectives and (abbreviated) learning trajectories of the associated courses?

Vector Line Integrals in Mathematics and Physics, IJRUME **9**, 92–117 (2023)

Concept Image

A *concept image* is the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes.

Experts have *rich* concept images,
which novices must accumulate gradually.

Tall & Vinner, Ed. Stud. Math. **12**, 151–169 (1981).

Learning Trajectory

A *learning trajectory* is a possible sequence of increasingly sophisticated understandings of a topic.

- Hypotheses about learning in a given domain;
- Include upper and lower anchors:
  - Upper anchor: goals for learning core knowledge and practices;
  - Lower anchor: ideas students bring to the classroom;
- Describe ways students may develop more sophisticated ways of thinking in a domain;
- Deepen the focus of science and mathematics education on central concepts rather than on inconsequential topics.

Duschl et al. [NRC] (2007); Lemke & Gonzales [NAGB] (2006); Plummer (2012)

Representational Transformation Diagram (RTD)

A flowchart to represent and analyze rich concept images.

Bajracharya, Emigh, and Manogue, Phys. Rev. Phys. Educ. Res. **15**, 020124 (2019).

- *Translation* (1 arrow in; 1 arrow out)
- *Consolidation* (≥ 1 arrows in)
- *Dissociation* (≥ 1 arrows out)

Length and complexity of RTD is proxy for cognitive load.

Look for:

- *Iconic expression or equation*;
- How the iconic expression is *unpacked*;
- What is the *starting point for calculation*.

Iconic expression: the symbolic representation of a fundamental concept in its simplest, most compact form.

- *Geometric* (independent of origin, coordinates, parameterization);
- *Easy to remember*;
- Contains instructions for *unpacking* in different contexts.

Calculus textbooks

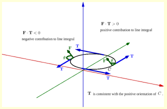
Calculus texbooks are written for two distinct audiences, students and instructors, with conflicting needs.

- Students read the book “backward from the homework problems,” then look at worked examples, but read the text itself only as a “last resort.”
- Instructors read textbooks “forward,” starting with the table of contents, checking that the desired topics are covered with the right level of rigor, and that there are enough problems.

McCallum (2001)

Riemann Sum

$$\int_C f \, ds$$



$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$ds = |\mathbf{r}'(t)| \, dt$$

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t) \, dt$$

$$\mathbf{F} = \langle f, g, h \rangle$$

$$dx = x'(t) \, dt, \ldots$$

$$d\mathbf{r} = \langle dx, dy, dz \rangle$$

$$\int_a^b \left(f(t)x'(t) + \ldots \right) \, dt$$

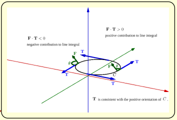
$$\int_C f \, dx + g \, dy + h \, dz$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Briggs, W., Cochran, L., Gillett, B., & Schulz, E. [3rd ed.] (2019)

Riemann Sum

$\int_C f\, ds$



$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$

$\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

$ds = |\mathbf{r}'(t)|\, dt$

$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt$

None of f, g, h, $d\vec{\mathbf{r}}$, ds, dx, dy, dz are used further...

Can solve HW algebraically.

$\mathbf{F} = \langle f, g, h \rangle$

$dx = x'(t)\, dt, \ldots$

$d\mathbf{r} = \langle dx, dy, dz \rangle$

$\int_a^b \left(f(t)x'(t) + \ldots \right)\, dt$

$\int_C f\, dx + g\, dy + h\, dz$

$\int_C \mathbf{F} \cdot d\mathbf{r}$

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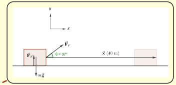
$$W = Fd$$



$$W = F_{\parallel} d = Fd\cos\theta$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$$

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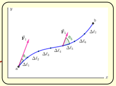
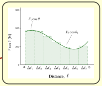


$$W \approx \sum F_i \cos\theta_i \Delta\ell_i$$



$$W = \int_a^b F\cos\theta\, d\ell$$

$$d\ell = |d\vec{\boldsymbol{\ell}}| \text{ (in words)}$$

$$W = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\ell}}$$

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + \ldots$$

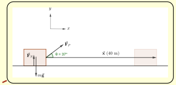
$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + \ldots$$

$$d\vec{\boldsymbol{\ell}} = dx\,\hat{\mathbf{i}} + \ldots$$

$$W = \int_{x_a}^{x_b} F_x\, dx + \ldots$$

Giancoli (2009)

$W = Fd$



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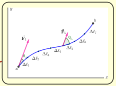

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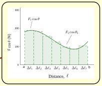

$W = \int_a^b F\cos\theta\, d\ell$

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$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + \ldots$

$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + \ldots$

$d\vec{\boldsymbol{\ell}} = dx\,\hat{\mathbf{i}} + \ldots$

$W = \int_{x_a}^{x_b} F_x\, dx + \ldots$

Problems use many representations.

Therefore start with iconic equation!

Giancoli (2009)

Definite Integrals

Way of thinking	Integrals are interpreted as …
Space underneath a graph	… the amount of space underneath the graph
Antiderivative	… an instruction to compute an antiderivative
Adding up pieces	… the summation of infinitesimal quantities
Accumulation from rate	… the accumulation from a rate function
Averaging	… an averaging across the domain
Procedural	… an operator to further a derivation

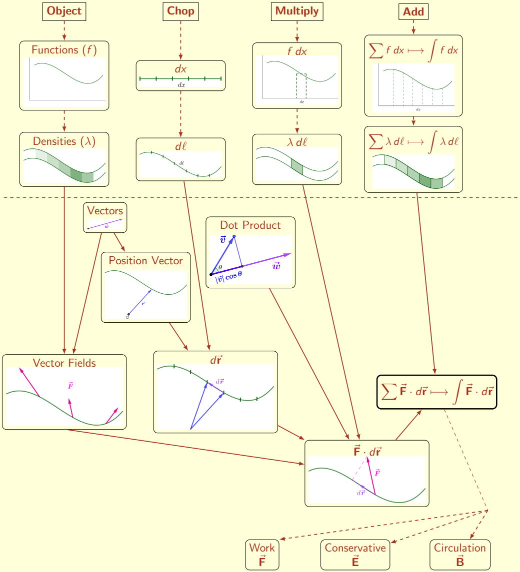
Adding up pieces $\longmapsto$ Multiplicatively-based summation

$\longmapsto$ Not yet sufficiently general

$\longmapsto$??

Jones (2013, 2015ab, 2020); Pina & Loverude (2019); Simmons & Oehrtman (2019, 2023)

Suggested learning trajectory



Chop, Multiply, Add

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https://bridge.math.oregonstate.edu/papers/IJRUMEintegrals.pdf

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