Octonions and the exceptional Lie algebras (and particle physics)

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Division Algebras

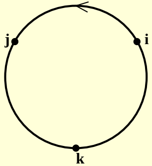
Real Numbers

$\mathbb{R}$

Quaternions

$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$

$q = (x + yi) + (r + si)j$



Complex Numbers

$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$

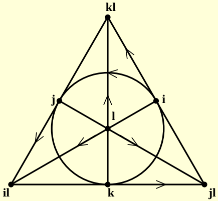
$z = x + yi$

Octonions

$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$

Split Octonions

$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$



$$I^2 = J^2 = -U, \; L^2 = +U$$

Split Division Algebras

$$I^2 = J^2 = -U,\ L^2 = +U$$

Signature (4, 4):

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

Definition

A *Lie Group* G is a group that is also a smooth manifold, and on which the group operations are smooth:

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (X, Y) & \longmapsto & X^{-1}Y \end{array}$$

Example

$$\begin{aligned} \mathrm{SO}(2) &= \{M \in \mathbb{R}^{2\times 2} | MM^T = I, \det M = 1\} \\ &= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \cong \mathbb{S}^1 \end{aligned}$$

continuous symmetry groups (rotations)

$|G| = \#$ of parameters

Definition

A *Lie algebra* is a vector space $\mathfrak{g}$ together with a binary operation

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (x, y) &\longmapsto [x, y] \end{aligned}$$

which is *bilinear* and satisfies

$$[x, y] = -[y, x]$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Example

$$\begin{aligned} \mathfrak{so}(3) &= \{A \in \mathbb{R}^{3\times 3} | A^t = -A, \mathrm{tr}(A) = 0\} \\ &= \langle r_x, r_y, r_z \rangle \end{aligned}$$

$$r_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{d}{d\theta}\bigg|_{\theta=0} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_x, r_y] = r_z$$

infinitesimal symmetries ($\mathfrak{g} = TG|_e$)
(WARNING: physicists use $-i\frac{d}{d\theta}$ to get Hermitian operators.)

$$|\mathfrak{g}| = \dim \mathfrak{g} = \dim TG = |G|$$

Representations

Definition

A *representation of a Lie group* G on a vector space V is a (group) homomorphism $\Pi : G \longmapsto GL(V)$.

Definition

A *representation of a Lie algebra* $\mathfrak{g}$ on a vector space V is a (Lie algebra) homomorphism $\rho : \mathfrak{g} \longmapsto \mathfrak{gl}(V)$.

(WARNING: The map ρ, the image matrices $\rho(\mathfrak{g})$, and the vector space V are all referred to as "representations of $\mathfrak{g}$", and similarly for G.)

Theorem

(Killing 1888–1890, Cartan 1894)
The ("simple") Lie groups are the classical groups

A_n	$SU(n+1)$
B_n	$SO(2n+1)$
C_n	$Sp(n)$
D_n	$SO(2n)$

together with the exceptional groups G_2, F_4, E_6, E_7, *and* E_8.

$$\begin{aligned} SU(n) &\equiv SU(n,\mathbb{C}) \\ SO(n) &\equiv SU(n,\mathbb{R}) \\ Sp(n) &\equiv SU(n,\mathbb{H}) \\ F_4 &\equiv SU(3,\mathbb{O}) \end{aligned}$$

$$\begin{aligned} G_2 &\equiv \mathrm{Aut}(\mathbb{O}) \\ E_6 &\equiv SL(3,\mathbb{O}) \\ E_7 &\equiv Sp(6,\mathbb{O}) \\ E_8 &\equiv ?? \end{aligned}$$

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$sa(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Description as symmetry groups

[Barton & Sudbery (2003), Wangberg (PhD 2007),
Dray & Manogue (CMUC 2010), Wangberg & Dray (JMP 2013, JAA 2014),
Dray, Manogue, & Wilson (CMUC 2014), Wilson, Dray, & Manogue (2022)]

The 2×2 Magic Square

Barton & Sudbery (2003):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

"Vinberg":

$$sa(2, \mathbb{A} \otimes \mathbb{B}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{A}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{B})$$

$$\mathfrak{so}(\mathrm{Im}\,\mathbb{H}) = \mathfrak{so}(3); \quad \mathfrak{so}(\mathrm{Im}\,\mathbb{O}) = \mathfrak{so}(7)$$

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014),
Dray, Huerta, & Kincaid (LMP 2014)]

$\mathfrak{so}(3)$

$$P = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

$$= x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

$$\Longrightarrow \det P = -(x^2 + y^2 + z^2)$$

Preserved by $P \longmapsto MPM^\dagger$ with $MM^\dagger = 1$. SU(2)!
Generated by $P \longmapsto AP + PA^\dagger$ with $A + A^\dagger = 0$. $\mathfrak{su}(2)$!
(Weyl) spinor: $\theta \in \mathbb{C}^2$, $\theta \longmapsto M\theta$.

$$s_x = -\frac{i}{2}\sigma_x \Longrightarrow [s_x, s_y] = s_z \Longrightarrow \mathfrak{su}(2) \cong \mathfrak{su}(3)$$

$$r_x^2 + r_y^2 + r_z^2 = -2I \text{ (spin 1), but } s_x^2 + s_y^2 + s_z^2 = -\tfrac{3}{4}I \text{ (spin } \tfrac{1}{2}\text{).}$$

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

group: $P \longmapsto MPM^\dagger$ algebra: $P \longmapsto AP + PA^\dagger$

$\mathfrak{so}(3,1)$

$$P = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (hermitian!): (so $P \longmapsto \{A, P\}$)

$$A = \sigma_x, \sigma_y, \sigma_z$$

$\mathfrak{so}(3,1)$

Vector in $\mathbb{C}' \oplus \mathbb{C}$

$$P = \begin{pmatrix} Lt + Uz & 1x - iy \\ 1x + iy & Lt - Uz \end{pmatrix}$$
$$= Lt\,\sigma_t + 1x\,\sigma_x + iy\,(-i\sigma_y) + Uz\,\sigma_z$$

Rotations (antihermitian!): (so $P \longmapsto [A, P]$)

$$A = i\sigma_x, i\sigma_y, i\sigma_z$$

Boosts (antihermitian!): (so $P \longmapsto [A, P]$)

$$X_L = L\sigma_x, \quad X_{iL} = L\sigma_y, \quad D_L = L\sigma_z$$

$$\boxed{\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C}' \otimes \mathbb{C})}$$

Guiding Principle #1

Lie algebras are real!

(signature matters)

$\mathfrak{so}(3,1)$ has boosts and rotations

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

$d = 3, 4, 6, 10$

(1980s: Corrigan, Evans, Fairlie, Manogue, Sudbery)

(1990s: Manogue & Schray)

Summary: 2×2 Magic Square

- The algebras in the 2×2 magic square are $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 2×2 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & p \\ -\overline{p} & 0 \end{pmatrix}$$

Idea: rotations/boosts acting on $\mathbb{K}' \oplus \mathbb{K}$:

$$D_i = D_{1i};\; D_L = D_{UL};\; X_i = X_{iU};\; X_L = X_{1L}$$

- The remaining basis elements are of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 \\ 0 & i \circ j \end{pmatrix} = \frac{1}{2}\left[D_i, D_j\right]$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$.

The 3 × 3 Magic Square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{sp}(6,\mathbb{C})$	$\mathfrak{sp}(6,\mathbb{H})$	$\mathfrak{sp}(6,\mathbb{O})$
$\mathbb{O}'$	??	??	??	??

Dray & Manogue (2010):
$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_{6(-26)} \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{Spin}(9,1)$

Dray, Manogue, & Wilson (2014): $E_7 \cong \mathrm{Sp}(6,\mathbb{O})$

Minimal representation of $\mathfrak{e}_8$ is adjoint!

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^\dagger \quad \Longrightarrow \quad P \longmapsto MPM^\dagger, \ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P], \ \theta \longmapsto A\theta$$

Idea: 2×2 vector and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6$, $\mathcal{P} \in H_3(\mathbb{O})$

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ -\theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{-1} \quad \Longrightarrow \quad P \longmapsto MPM^{-1},\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: 2×2 adjoint and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6, \mathcal{P} \in \mathfrak{e}_6$ (3×3 adjoint action!)

Summary: 3×3 Magic Square

- The algebras in the 3×3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 3×3 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}, \quad X_p = \begin{pmatrix} 0 & p & 0 \\ -\overline{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\overline{p} & 0 \end{pmatrix}, \quad Z_p = \begin{pmatrix} 0 & 0 & -\overline{p} \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}$$

- The remaining basis elements can be chosen to be of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 & 0 \\ 0 & i \circ j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$. **TRIALITY!**

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!
- $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \mathbf{128}$.
- The **128** is a Majorana–Weyl spinor rep of $\mathfrak{so}(12,4)$.
- The **128** contains spinor reps of each 2×2 algebra.

Particle Physics

Fundamental particles (leptons and quarks) are Lorentz ($\mathfrak{so}(3,1)$) *spinors*, and carry representations of electromagnetism ("charge"; $\mathfrak{u}(1)$), the weak interaction ($\mathfrak{su}(2)_L$), and the strong interaction ("color"; $\mathfrak{su}(3)$).

Mediators (photons, vector bosons, gluons) are Lorentz *vectors*, and carry (adjoint?) representations of the interactions.

Want simultaneous representations of the Lorentz group $SO(3,1)$ and the Standard Model group $U(1) \times SU(2)_L \times SU(3)$.

Guiding Principle #3

All representations live in $\mathfrak{e}_8$!

$$
\begin{aligned}
\mathfrak{e}_{8(-24)} &= \mathfrak{so}(12,4) + \text{spinors} \\
\mathfrak{so}(12,4) &\supset \mathfrak{so}(3,1) + \mathfrak{so}(7,3) + \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) + \mathfrak{so}(4) + \mathfrak{so}(3,3) + \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) + \mathfrak{su}(2)_L + \mathfrak{su}(2)_R + \mathfrak{su}(3)_c + \mathfrak{u}(1) + \mathfrak{so}(2)
\end{aligned}
$$

SUMMARY

Lie algebras are real!
The 3 × 3 structure is broken to 2 × 2.
All representations live in $\mathfrak{e}_8$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \text{spinors}$$

$$\mathfrak{so}(12,4) \supset \mathfrak{so}(3,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \text{``} \otimes \mathbb{C}\text{''}$$

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