OCTONIONS and FERMIONS

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I: Octonions

II: Dimensional Reduction

III: Leptons

IV: Cayley Spinors

DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$

$$z = x + yi$$

Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$

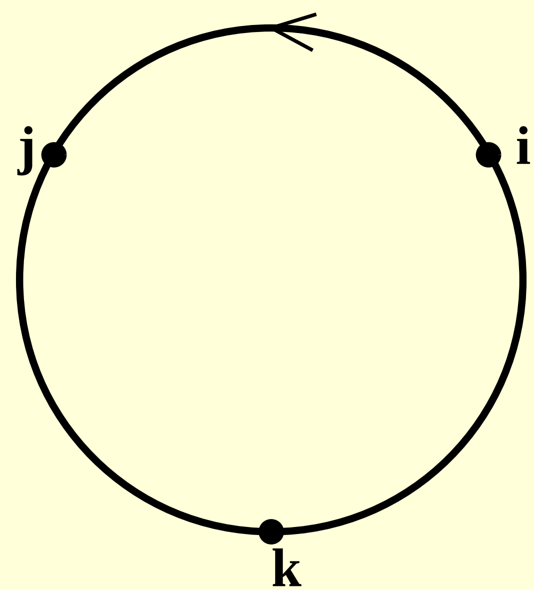
$$q = (a + bi) + (c + di)j$$

$$i^2 = j^2 = -1$$

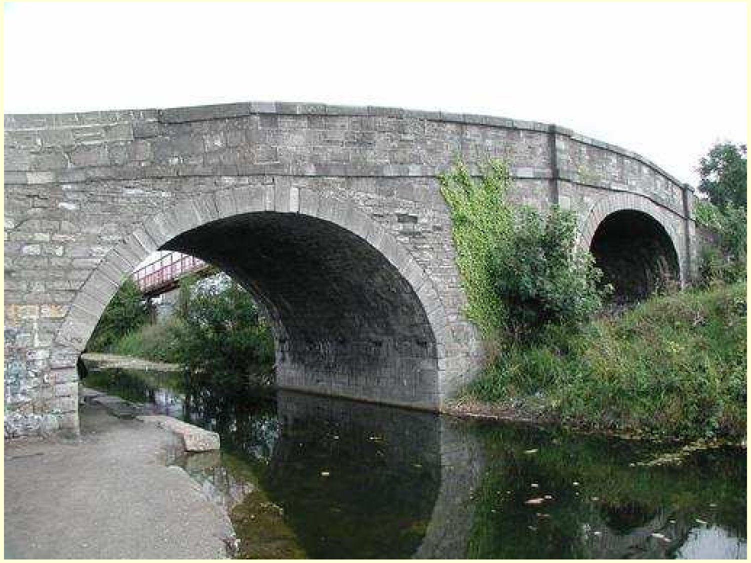
QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

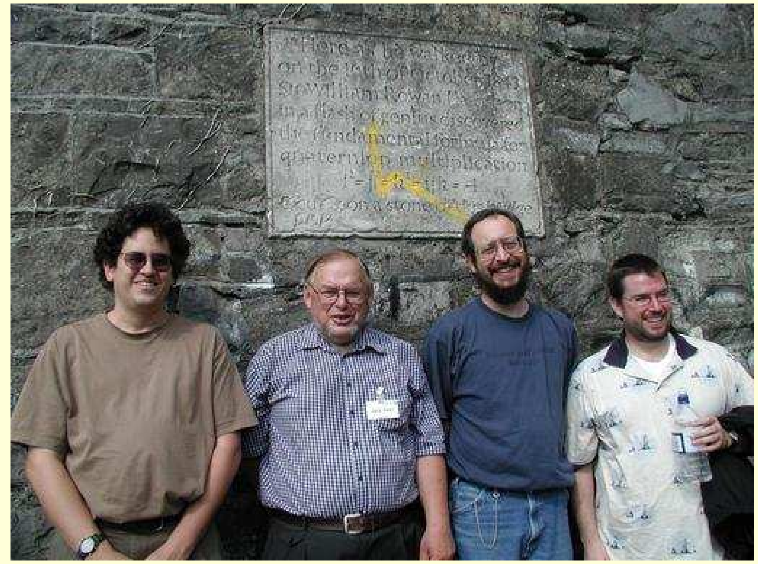
not commutative



THE DISCOVERY OF THE QUATERNIONS



Brougham Bridge (Dublin)



Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

VECTORS I

$$v = bi + cj + dk \longleftrightarrow \vec{\boldsymbol{v}} = b\,\hat{\boldsymbol{\imath}} + c\,\hat{\boldsymbol{\jmath}} + d\,\hat{\boldsymbol{k}}$$

$$vw \longleftrightarrow -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$$

Dot product exists in any dimension
but
Cross product exists only in 3 and 7 dimensions

DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$

$$q = (a + bi) + (c + di)j$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$

$$z = x + yi$$

Octonions:

$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

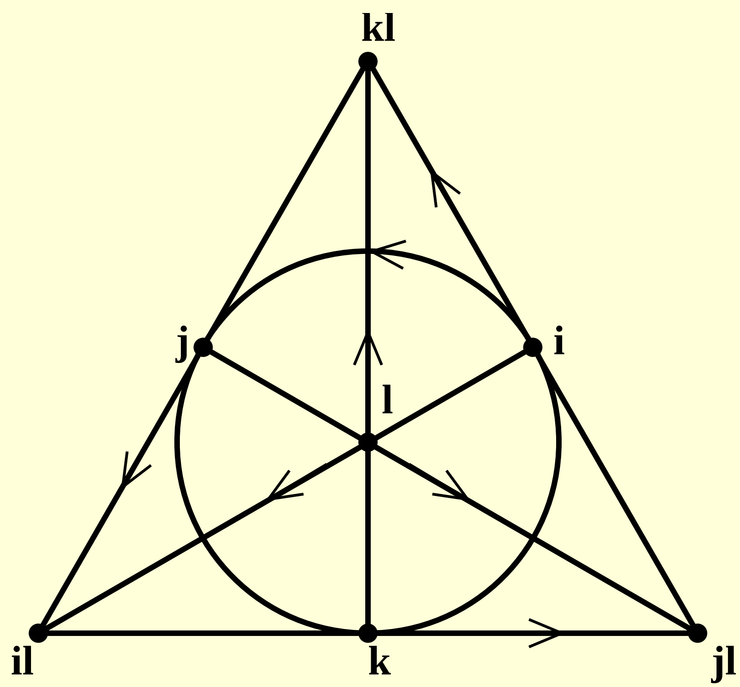
$$i^2 = j^2 = \ell^2 = -1$$

OCTONIONS

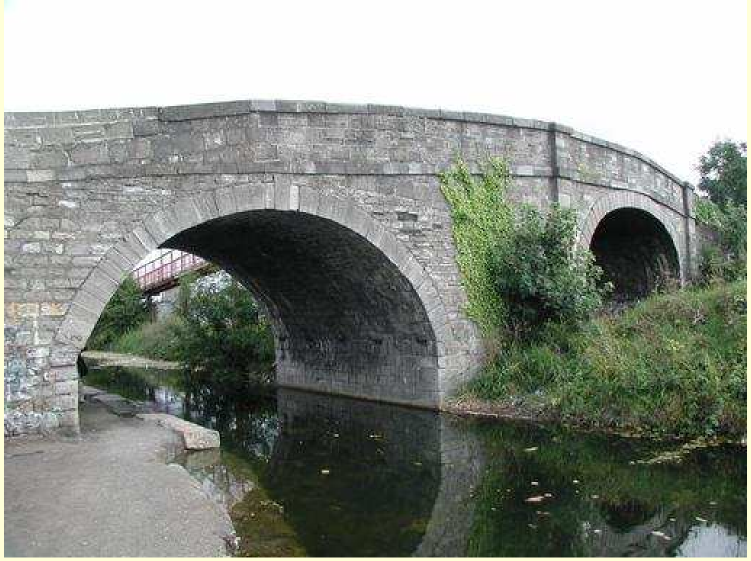
each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative



THE DISCOVERY OF THE OCTONIONS



Brougham Bridge (Dublin)

John T. Graves (1843!)
Arthur Cayley (1845)
octaves, Cayley numbers

VECTORS II

$$\boldsymbol{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$-\det(\boldsymbol{X}) = -t^2 + x^2 + y^2 + z^2$$

- {vectors in (3+1)-dimensional spacetime}
  $\longleftrightarrow$ {2×2 complex Hermitian matrices}
- determinant $\longleftrightarrow$ (Lorentzian) inner product

LORENTZ TRANSFORMATIONS

Exploit (local) isomorphism:

$$SO(3,1) \approx SL(2,\mathbb{C})$$

$$\boldsymbol{x}' = \boldsymbol{\Lambda x} \qquad \longleftrightarrow \qquad \boldsymbol{X}' = \boldsymbol{M X M}^\dagger$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\det(\boldsymbol{M}) = 1 \quad \Longrightarrow \quad \det \boldsymbol{X}' = \det \boldsymbol{X}$$

WHICH DIMENSIONS?

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \longmapsto$

$$\boldsymbol{X} = \begin{pmatrix} p & \overline{a} \\ a & m \end{pmatrix} \qquad (p, m \in \mathbb{R}; a \in \mathbb{K})$$

$\dim \mathbb{K} + 2 = 3, 4, 6, 10$ spacetime dimensions

supersymmetry

$$SO(5,1) \approx SL(2, \mathbb{H})$$
$$SO(9,1) \approx SL(2, \mathbb{O})$$

ROTATIONS

$$\boldsymbol{M}_{zx} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\boldsymbol{M}_{xy} = \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} \qquad \boldsymbol{M}_{yz} = \begin{pmatrix} \cos\frac{\alpha}{2} & -i\ \sin\frac{\alpha}{2} \\ -i\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$i \longrightarrow j, k, ..., \ell$$

$\mathbb{H}$: $\boldsymbol{M} = e^{k\theta}\boldsymbol{I}$

$\mathbb{O}$: need flips

$$\boldsymbol{X}' = \boldsymbol{M}_2(\boldsymbol{M}_1\boldsymbol{X}\boldsymbol{M}_1^\dagger)\boldsymbol{M}_2^\dagger$$

$$\boldsymbol{M}_1 = i\boldsymbol{I} \qquad \boldsymbol{M}_2 = (i\cos\theta + j\sin\theta)\boldsymbol{I}$$

nesting

PENROSE SPINORS

$$v = \begin{pmatrix} c \\ \overline{b} \end{pmatrix}$$

$$vv^\dagger = \begin{pmatrix} |c|^2 & cb \\ \overline{b}\overline{c} & |b|^2 \end{pmatrix}$$

$$\det(vv^\dagger) = 0$$

$(\text{spinor})^2 = \text{null vector}$

Lorentz transformation:

$$v' = \boldsymbol{M} v$$

$$\boldsymbol{M}(vv^\dagger)\boldsymbol{M}^\dagger = (\boldsymbol{M}v)(\boldsymbol{M}v)^\dagger$$

compatibility

WEYL EQUATION

- **Massless, relativistic, spin $\frac{1}{2}$**
- **Momentum space**

$$\widetilde{\boldsymbol{P}}\psi = 0$$

$$\widetilde{\boldsymbol{P}} = \boldsymbol{P} - (\mathrm{tr}\,\boldsymbol{P})\,\boldsymbol{I}$$

$$\Longrightarrow \det(P) = 0 \qquad \text{(3 of 4 string equations!)}$$

One solution: $(P, \theta$ complex$)$

$$P = \pm\theta\theta^\dagger$$

$$\widetilde{\theta\theta^\dagger}\theta = (\theta\theta^\dagger - \theta^\dagger\theta)\theta = \theta\theta^\dagger\theta - \theta^\dagger\theta\theta = 0$$

General solution: $(\xi \in \mathbb{O})$

$$\psi = \theta\xi$$

P, ψ quaternionic

DIRAC EQUATION

4×4 complex:

$$0 = (\gamma_t \gamma_\mu \, p^\mu - m \, \gamma_t) \, \Psi$$

2×2 quaternionic:

$$\begin{aligned} 0 &= \left(p^t \sigma_t - p^\alpha \sigma_\alpha - m \, \sigma_k\right) \psi \\ &= -\widetilde{P} \psi \end{aligned}$$

Isomorphism: $(\mathbb{H}^2 \approx \mathbb{C}^4)$

$$\begin{pmatrix} c - kb \\ d + ka \end{pmatrix} \longleftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

DIMENSIONAL REDUCTION

$$SO(3,1) \approx SL(2,\mathbb{C}) \subset SL(2,\mathbb{O}) \approx SO(9,1)$$

Projection: $(\mathbb{O} \to \mathbb{C})$

$$\pi(p) = \frac{1}{2}(p + \ell p \bar{\ell})$$

Determinant: $\det(P) = 0 \Longrightarrow$

$$\det\big(\pi(P)\big) = m^2$$

Mass Term: $P = \pi(P) + m\,\sigma_k$ $\qquad \sigma_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$

$$P = \begin{pmatrix} p^t + p^z & p^x - \ell p^y - km \\ p^x + \ell p^y + km & p^t - p^z \end{pmatrix}$$

SPIN

Finite rotation:

$$R_z = \begin{pmatrix} e^{\ell\frac{\theta}{2}} & 0 \\ 0 & e^{-\ell\frac{\theta}{2}} \end{pmatrix}$$

Infinitesimal rotation:

$$L_z = \left.\frac{dR_z}{d\theta}\right|_{\theta=0} = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

Right self-adjoint operator:

$$\hat{L}_z\psi := (L_z\psi)\bar{\ell}$$

Right eigenvalue problem:

$$\hat{L}_z\psi = \psi\lambda$$

ANGULAR MOMENTUM REVISITED

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \qquad L_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$L_z = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix} \qquad \hat{L}_\mu \psi := -(L_\mu \psi)\ell$$

$$\psi = e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow$$

$$\hat{L}_z \psi = \psi \, \frac{1}{2} \qquad \hat{L}_x \psi = -\psi \, \frac{k}{2} \qquad \hat{L}_y \psi = -\psi \, \frac{k\ell}{2}$$

Simultaneous eigenvector!

(only 1 *real* eigenvalue)

LEPTONS

ψ | $P = \psi\psi^\dagger$

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad \nu_{-z} \nu_{-z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

How Many Quaternionic Spaces?

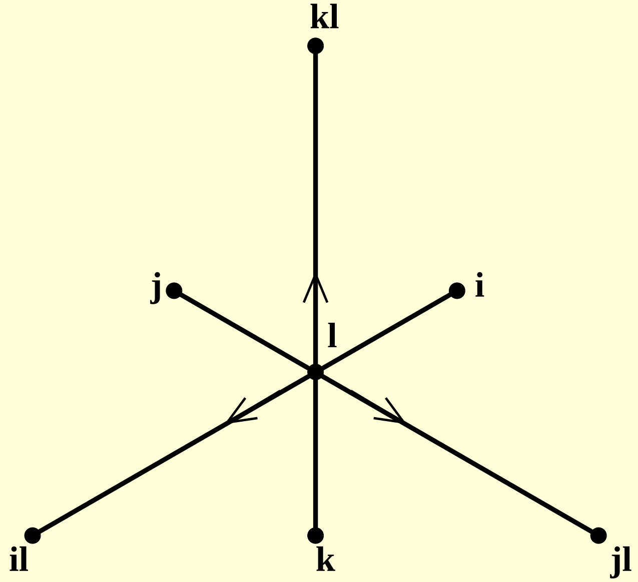
Dimensional Reduction:

$$\Longrightarrow \ell \in \mathbb{H}$$

Orthogonality:

$$(\mathbb{H}_1 \cap \mathbb{H}_2 = \mathbb{C})$$

$$\longmapsto i, j, k$$



Answer: $3!$

LEPTONS

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix} \qquad \nu_{-z} \nu_{-z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\emptyset_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \emptyset_z \emptyset_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

WHAT NEXT?

Have:

3 generations of leptons!

Neutrinos have just one helicity!

What about $\emptyset_z$?

Want:

- interactions
- quarks/color ($SU(3)$!)
- charge

JORDAN ALGEBRAS

Exceptional quantum mechanics:

(Jordan, von Neumann, Wigner)

$$(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{X}^2 = \mathcal{X} \circ \left(\mathcal{Y} \circ \mathcal{X}^2\right)$$

$$\mathcal{X} = \begin{pmatrix} p & a & \overline{c} \\ \overline{a} & m & b \\ c & \overline{b} & n \end{pmatrix}$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I} \end{aligned}$$

EXCEPTIONAL GROUPS

$\boldsymbol{F_4}$: "$SU(3,\mathbb{O})$"

$$(\mathcal{M}\mathcal{X}\mathcal{M}^\dagger)\circ(\mathcal{M}\mathcal{Y}\mathcal{M}^\dagger) = \mathcal{M}(\mathcal{X}\circ\mathcal{Y})\mathcal{M}^\dagger$$

$\boldsymbol{E_6}$: "$SL(3,\mathbb{O})$"

$$\det\mathcal{X} = \frac{1}{3}\operatorname{tr}\left((\mathcal{X}*\mathcal{X})\circ\mathcal{X}\right)$$

$$SO(3,1)\times U(1)\times SU(2)\times SU(3)\subset E_6$$

$$\mathcal{X} = \begin{pmatrix} \boldsymbol{X} & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \mathcal{M}\mathcal{X}\mathcal{M}^\dagger = \begin{pmatrix} \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^\dagger & \boldsymbol{M}\theta \\ \theta^\dagger\boldsymbol{M}^\dagger & n \end{pmatrix}$$

Supersymmetry

CAYLEY SPINORS

$$\mathcal{P} = \begin{pmatrix} \boldsymbol{P} & \theta\xi \\ \overline{\xi}\theta^\dagger & |\xi|^2 \end{pmatrix}$$

$$\mathcal{P} * \mathcal{P} = 0 \Longrightarrow \widetilde{\boldsymbol{P}}\theta = 0$$

$$\Longrightarrow \mathcal{P} = \psi\psi^\dagger$$

quaternionic!

Furthermore: $$\mathcal{X}^\dagger = \mathcal{X} \Longrightarrow \mathcal{X} = \sum_{n=1}^{3} \lambda_n \psi_n \psi_n^\dagger$$

leptons, mesons, baryons?

DISCUSSION

- 1-squares describe spin/helicity of leptons.
- Three generations of particles (& sterile neutrino).
- 3+1 space-time emerges from symmetry-breaking.
- Do 2-squares and 3-squares describe mesons and baryons?
- Does broken E_6 describe standard model?
- Do discrete group transformations yield charge?

Life is complex.

It has real and imaginary parts.

Life is octonionic...

THE END