

Calculus Meets Physics:  
*25 years of interdisciplinary discussion*

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# The Paradigms in Physics Project

- Complete redesign of physics major – 20 new courses
- Junior-year “paradigms” designed around common themes.
- Senior-year “capstones” finish traditional disciplinary content.
- 25+ years of continuous NSF funding.
- Living curriculum: *Monthly curriculum meetings for 25+ years!*
- Paradigms 2.0 implemented in 2017.
- Active engagement: *300+ documented activities!*

<https://paradigms.oregonstate.edu>



# Mathematics vs. Physics

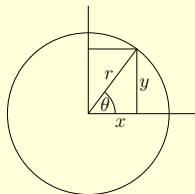
Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where  $k$  is a constant. What is  $T(r, \theta)$ ?

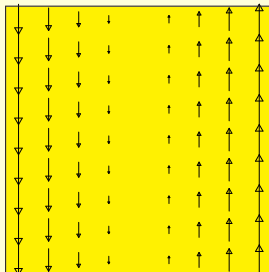
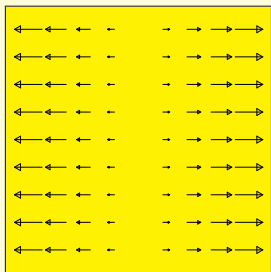
**A:**  $T(r, \theta) = kr^2$

**B:**  $T(r, \theta) = k(r^2 + \theta^2)$



**Mathematics and Physics are two disciplines  
separated by a common language!**

# Geometric Reasoning



- Which vector field is conservative?
- Which vector field has nonzero curl?
- Which vector field has nonzero divergence?

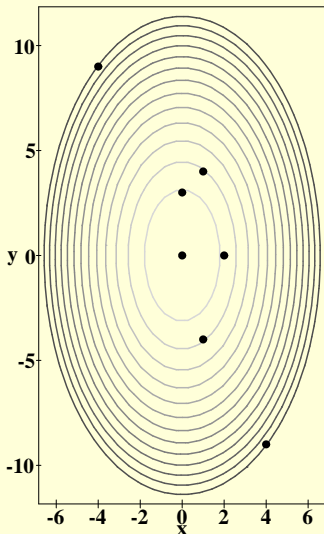
Which vector field could represent an electric field? a magnetic field?

# The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates  $(x, y)$  measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height  $h$  of the hill in feet above sea level is given by

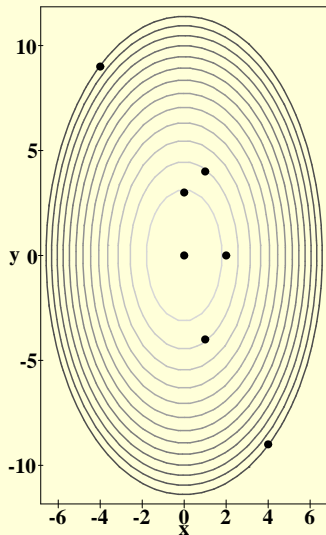
$$h = a - bx^2 - cy^2$$

where  $a = 5000$  ft,  $b = 30 \frac{\text{ft}}{\text{mi}^2}$ ,  
and  $c = 10 \frac{\text{ft}}{\text{mi}^2}$ .

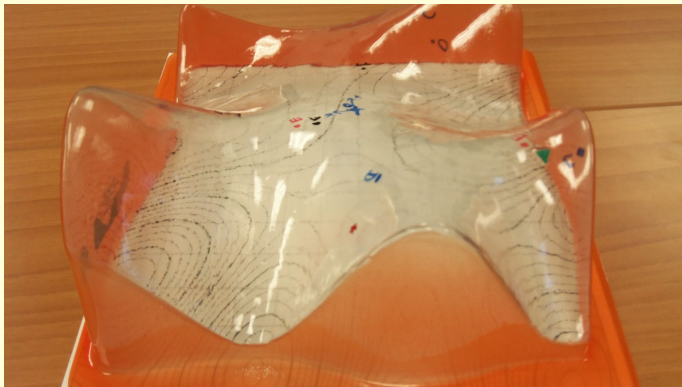


# Kinesthetic Activity

*Stand up and close your eyes.  
Hold out your right arm in the  
direction of the gradient where  
you are standing.*



# Surfaces



(Each surface is dry-erasable, as are the matching contour maps.)

*Raising Calculus to the Surface* (Aaron Wangberg)

*Raising Physics to the Surface* (+ Liz Gire, Robyn Wangberg)

<https://raisingcalculus.winona.edu>



## State Variables:

$T$  = temperature

$S$  = entropy

$p$  = pressure

$V$  = volume

## First Law:

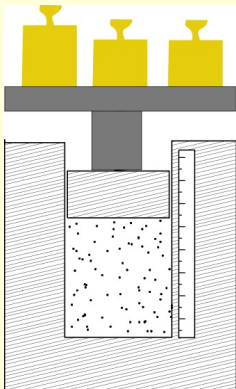
( $U$  = internal energy)

$$dU = T dS - p dV$$

- Compressibility =  $-\frac{1}{V} \frac{\partial V}{\partial p}$
- Design an experiment to measure compressibility.



# Name the Experiment



$$-\frac{1}{V} \frac{\partial V}{\partial p}$$

- $\left(\frac{\partial p}{\partial T}\right)_V \neq \left(\frac{\partial p}{\partial T}\right)_S$ .
- What are the independent variables??
- Can not hold “everything else” fixed!

← What is this material??

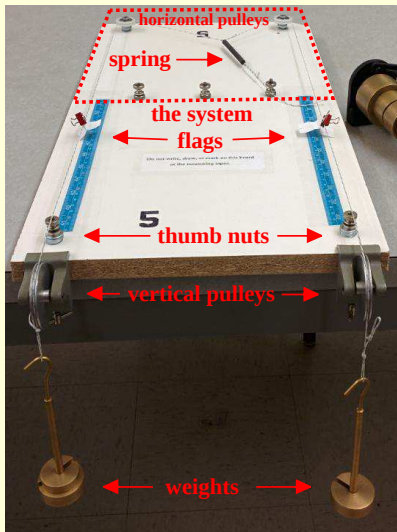
Isothermal:  $-\frac{1}{V} \left(\frac{\partial V}{\partial p}\right)_T$

Isoentropic:  $-\frac{1}{V} \left(\frac{\partial V}{\partial p}\right)_S$

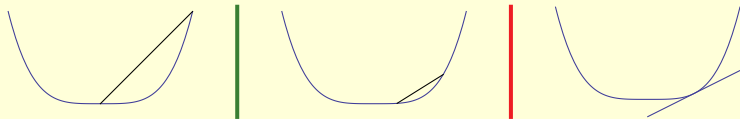
# Partial Derivative Machine

- Developed for junior-level thermodynamics course
- Two positions,  $x_i$ , two string tensions (masses),  $F_i$ .
- “Find  $\frac{\partial x}{\partial F}$ .”
- Idea: Measure  $\Delta x$ ,  $\Delta F$ ; divide.
- Mathematicians:  
“That’s not a derivative!”

Roundy et al., *Experts’ Understanding of Partial Derivatives Using the Partial Derivative Machine*, PERC 2014



# Thick Derivatives



**Math:**  $\exists$  “bright line” between *average* rate of change and *instantaneous* rate of change.

(Such averages are used to approximate derivatives.)

**Physics:** “Average” refers to secant lines, not (good) approximations to tangent lines.

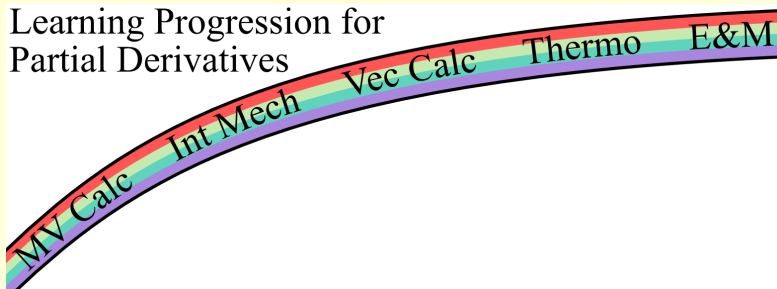
**Move the bright line!**

**Thick Derivatives!**

(Derivatives are fundamentally ratios of small changes, not limits.)

[Dray, AMS Blog on Education, 5/31/16]

# Learning Progression

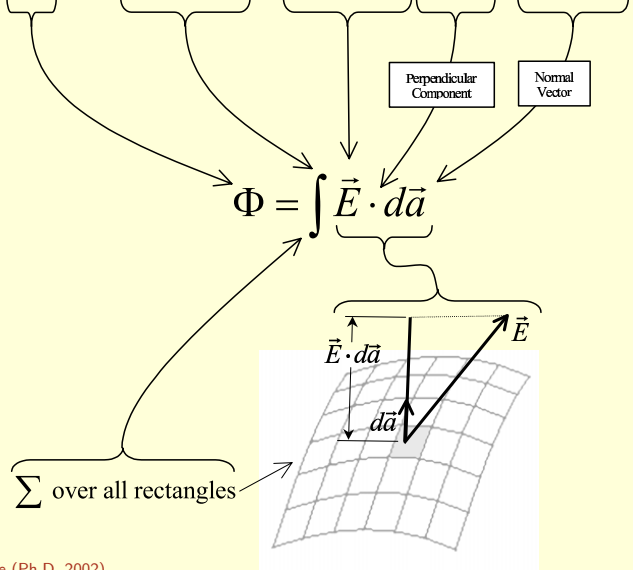


- Successively more sophisticated ways of thinking about a topic.
- Sequences supported by research on learner's ideas and skills.
- *Lower anchor* grounded in students' prior ideas and skills.
- *Upper anchor* grounded in knowledge and practices of experts.

Duschle et al., NRC, 2007; Plummer, 2012; Sikorski et al., 2009, 2010  
Manogue, Dray, Emigh, Gire, & Roundy, PERC 2017


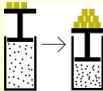

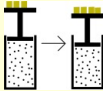

# Multiple Representations

Flux is the total amount of electric field through a given area.



Kerry Browne (Ph.D. 2002)

# Extended Theoretical Framework for Concept of Derivative

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		“avg. rate of change”	$\frac{f(x+\Delta x) - f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		“inst. rate of change”	$\lim_{\Delta x \rightarrow 0} \dots$	...with $\Delta x$ small	
Function		“...at any point/time”	$f'(x) = \dots$	... depends on $x$	tedious repetition

**No entry for symbolic differentiation!!**

Roundy, Dray, Manogue, Wagner, & Weber, CRUME 18 Proceedings, MAA, 2015. <https://sigmaa.maa.org/rume/Site/Proceedings.html>

# Differentials

Does  $\frac{df}{dx}$  mean “ $f'(x)$ ” or “ $df$  over  $dx$ ”?

$$d(u^2) = 2u \, du$$

$$d(\sin u) = \cos u \, du$$

**Instead of:**

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

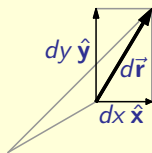
**One coherent idea:**

“Zap equations with  $d$ ”

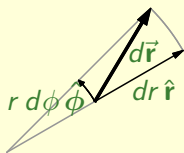
(infinitesimal reasoning)

Dray & Manogue, CMJ **34**, 283–290 (2003); CMJ **41**, 90–100 (2010).

## Vector calculus is about one coherent concept: Infinitesimal Displacement



$$d\vec{r} = dx \hat{x} + dy \hat{y}$$



$$d\vec{r} = dr \hat{r} + r d\phi \hat{\phi}$$

$$\begin{aligned} ds &= |d\vec{r}| \\ d\vec{A} &= d\vec{r}_1 \times d\vec{r}_2 \\ dA &= |d\vec{r}_1 \times d\vec{r}_2| \\ dV &= (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3 \end{aligned}$$



# SUMMARY

- Use multiple representations, including geometry, measurement, numerical data;
- Always ask both “With respect to what,” **and** “With what held constant.”



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<https://math.oregonstate.edu/bridge>  
<https://books.physics.oregonstate.edu/GVC>  
<https://paradigms.oregonstate.edu>  
<https://raisingcalculus.winona.edu>