# Partial Derivatives
## in Calculus and Upper-Level Physics Courses

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- Mount Holyoke College
- Utah State University

# Dissemination of Curriculum

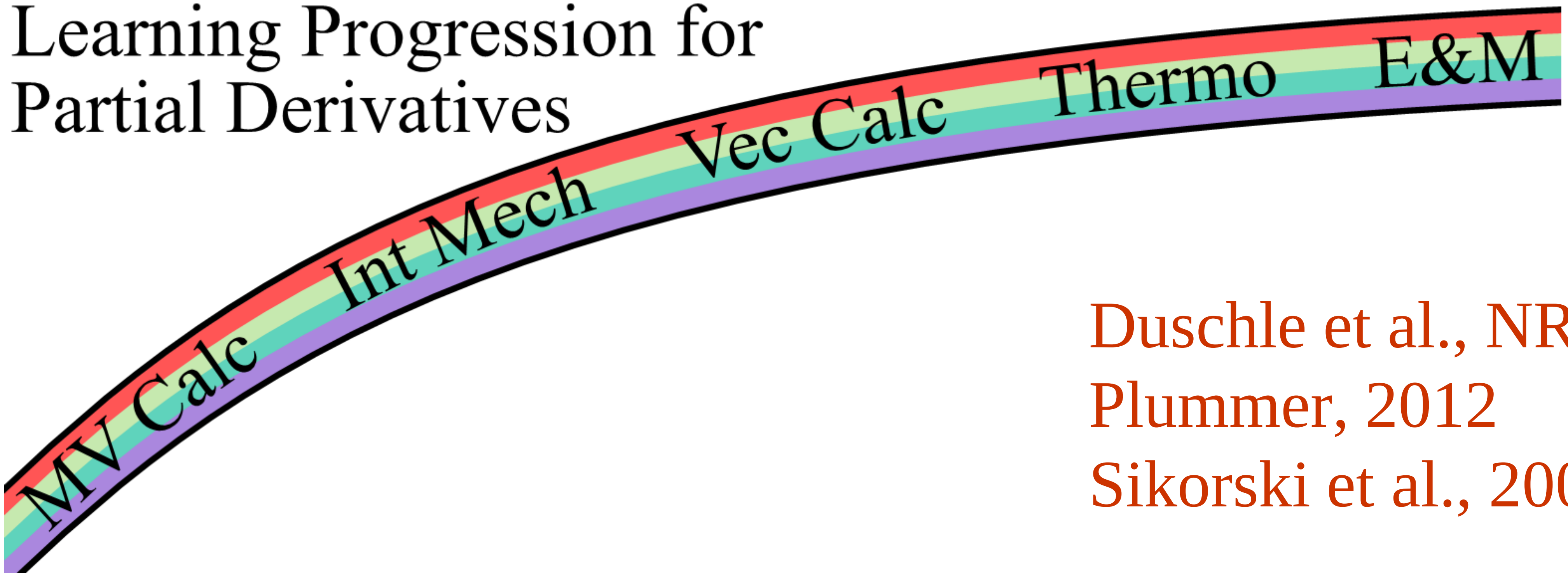
- Old: Textbook authors determined order. lecture, reading, homework
- Now: Who determines the order?

  in-class activities, SWBQs/concept tests, mini-lectures, video, online short readings, flipping and backflipping, …
- Challenge: How do we disseminate 2 decades of new holistic curriculum structures in the era of active engagement/online resources?

# Learning Progressions

- Successively more sophisticated ways of thinking about a topic.
- Sequences that are supported by research on learner's ideas and skills.

Learning Progression for
Partial Derivatives




Duschle et al., NRC, 2007
Plummer, 2012
Sikorski et al., 2009, 2010

# Learning Progressions

- What is an effective content sequence?
- Different types of resources:  activities, SWBQs, text bits, homework problems, …
- What research supports these choices?

# Learning Progressions

- Lower anchor grounded in prior ideas and skills students bring to the classroom.
- Upper anchor grounded in knowledge and practices of experts.

# What is a Concept Image?

- Concept Image: the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes.

Tall and Vinner, *Educ. Stud. Math.*, (1981).

# Small White Board Questions (SWBQs)

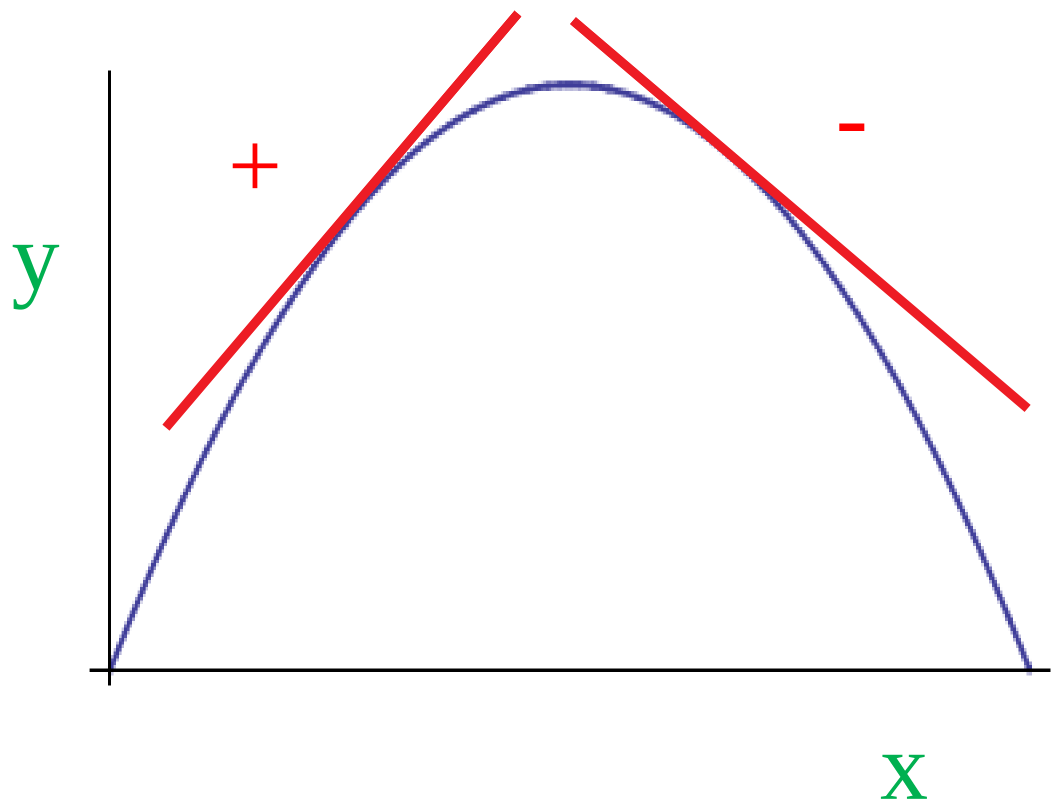
- For this audience:
  - Write an element of your concept image of derivative.
- For students:
  - Write something that you know about derivatives.

# Concept Image of Derivative

- Ratio
- Slope
- Limit
- Function
- Rate of Change
- Velocity
- Difference Quotient

# Lower Anchor for Derivatives

$$y = f(x)$$



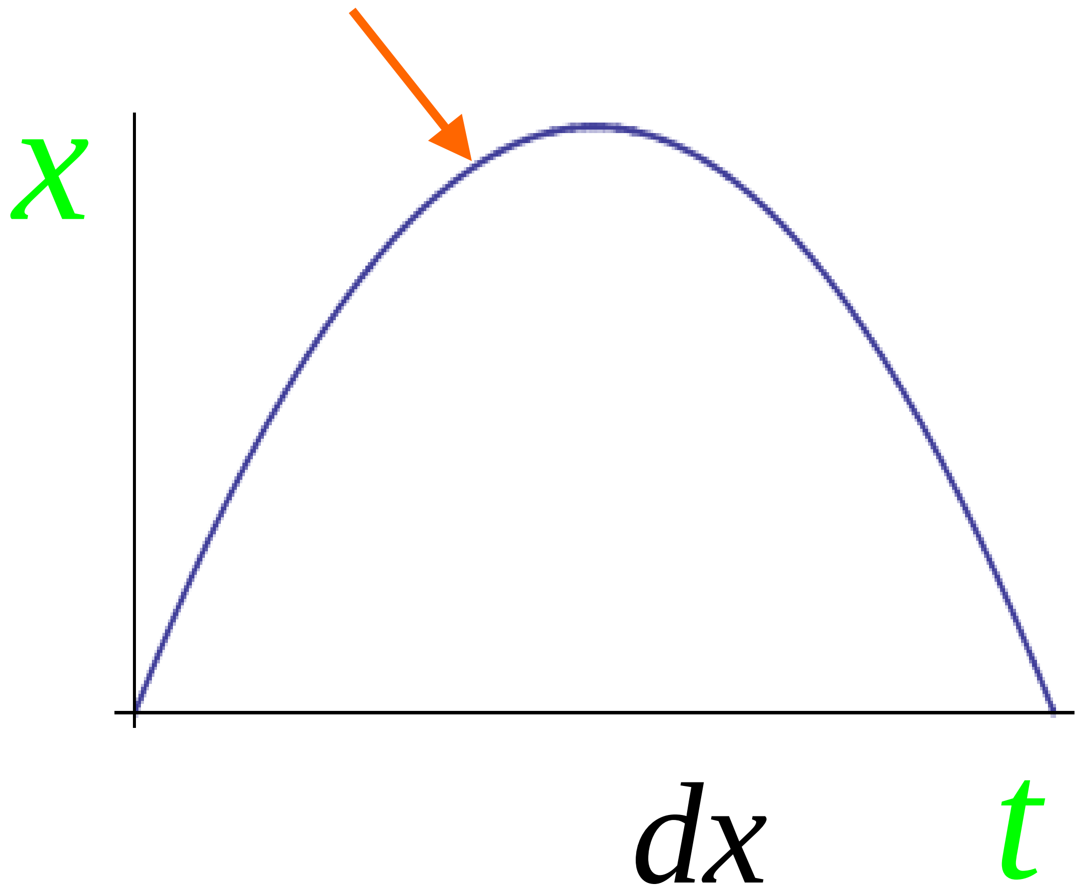

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$(\cos x)' = -\sin x$$

Derivative is slope of tangent line.

# Mechanics—Lower Division

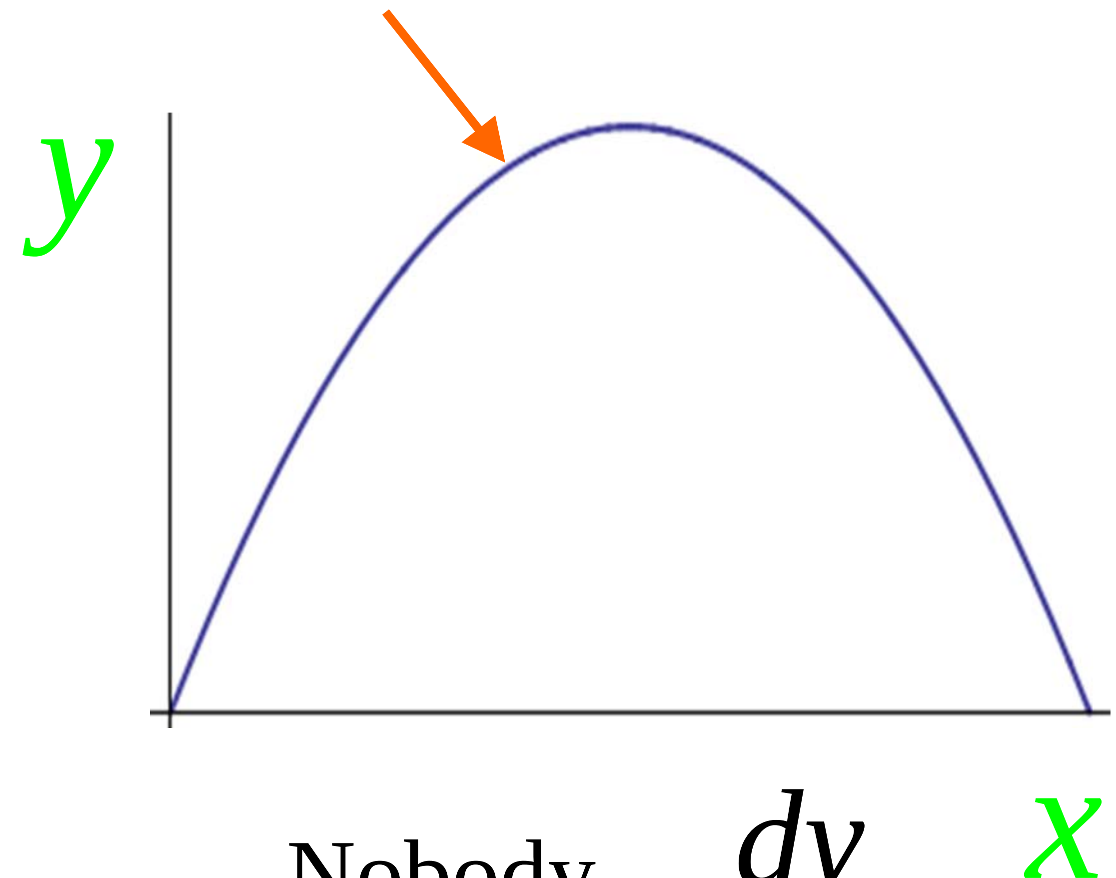
This is a Function




$$v = \frac{dx}{dt}$$

Derivative = Speed=Slope

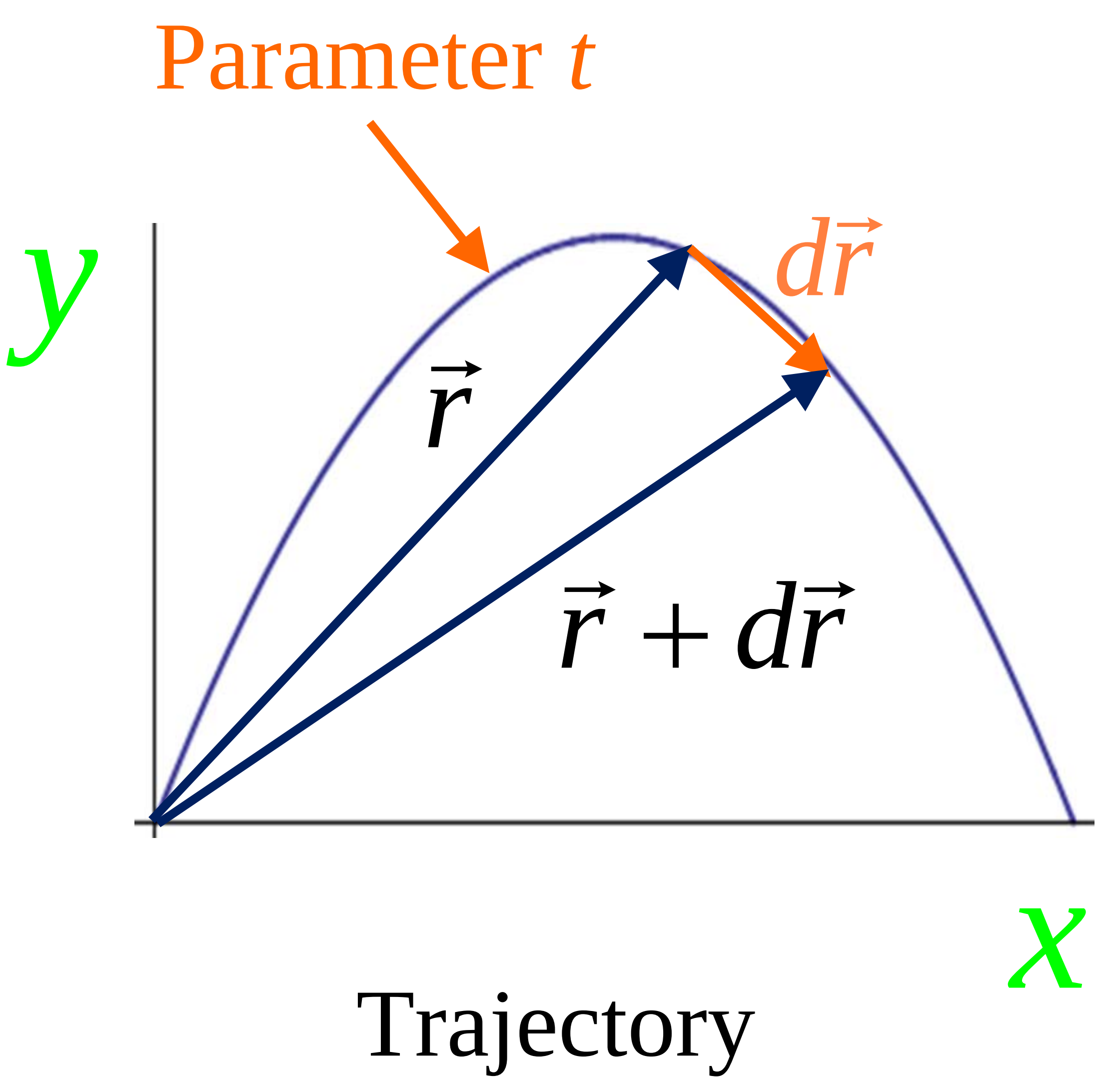
This is a Trajectory




$$\text{Nobody cares} = \frac{dy}{dx}$$

Derivative = Slope

# Mechanics—Upper Division

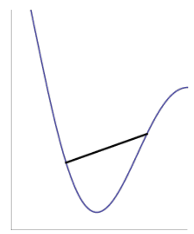
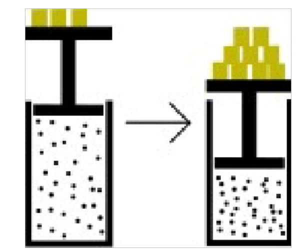
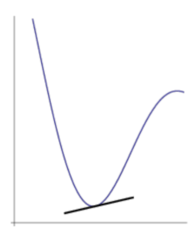
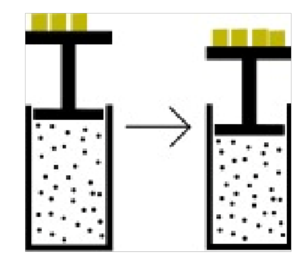
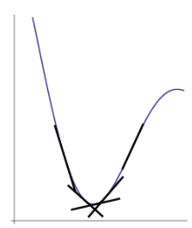



$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y}$$

- Speed is NOT slope.
- Velocity points in direction of slope.

# Extended Framework

| Process-object layer | Graphical | Verbal | Symbolic | Numerical | Physical |
|---|---|---|---|---|---|
| | Slope | Rate of Change | Difference Quotient | Ratio of Changes | Measurement |
| Ratio |  | "avg. rate of change" | $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ | $\frac{y_2-y_1}{x_2-x_1}$ numerically |  |
| Limit |  | "inst. rate of change" | $\lim_{\Delta x \to 0} \cdots$ | ...with $\Delta x$ small |  |
| Function |  | "...at any point/time" | $f'(x) =$ ... | ... depends on $x$ | tedious repetition |

| Process-object layer | · · Symbolic · · |
|---|---|
| | Instrumental Understanding |
| Function | *rules to "take a derivative"* |

Zandieh, CBMS Issues in Math Ed, 2000.

Roundy, et al., RUME, 2015.

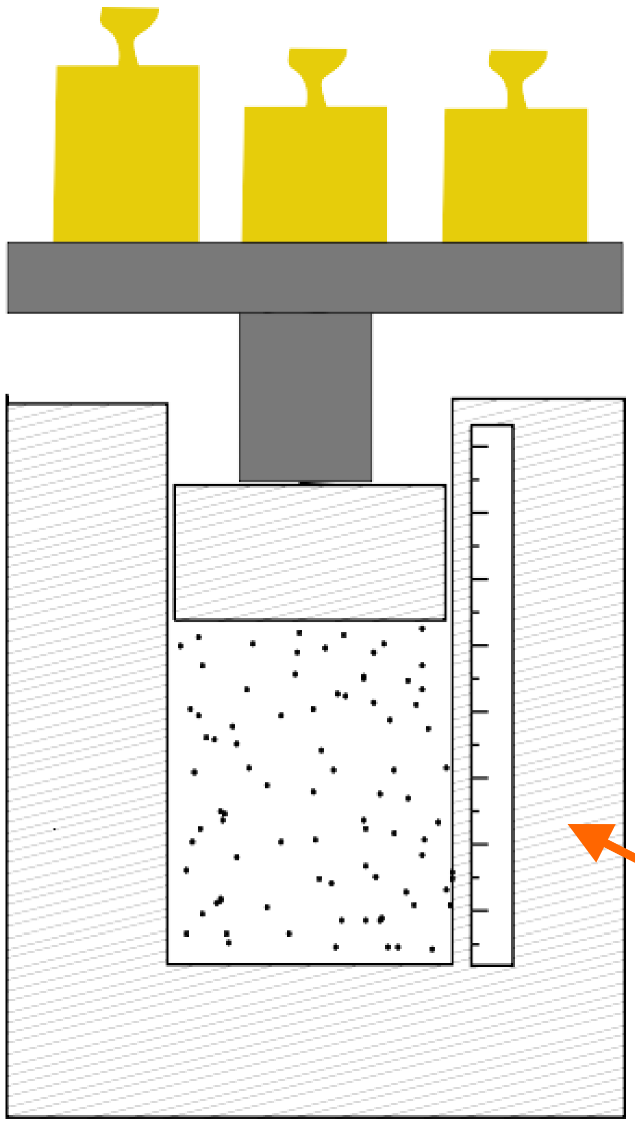
# Name the Experiment

- Design an experiment to measure compressibility:

$$\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \quad vs. \quad \beta_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$$

Isothermal Isentropic

# Name the Experiment

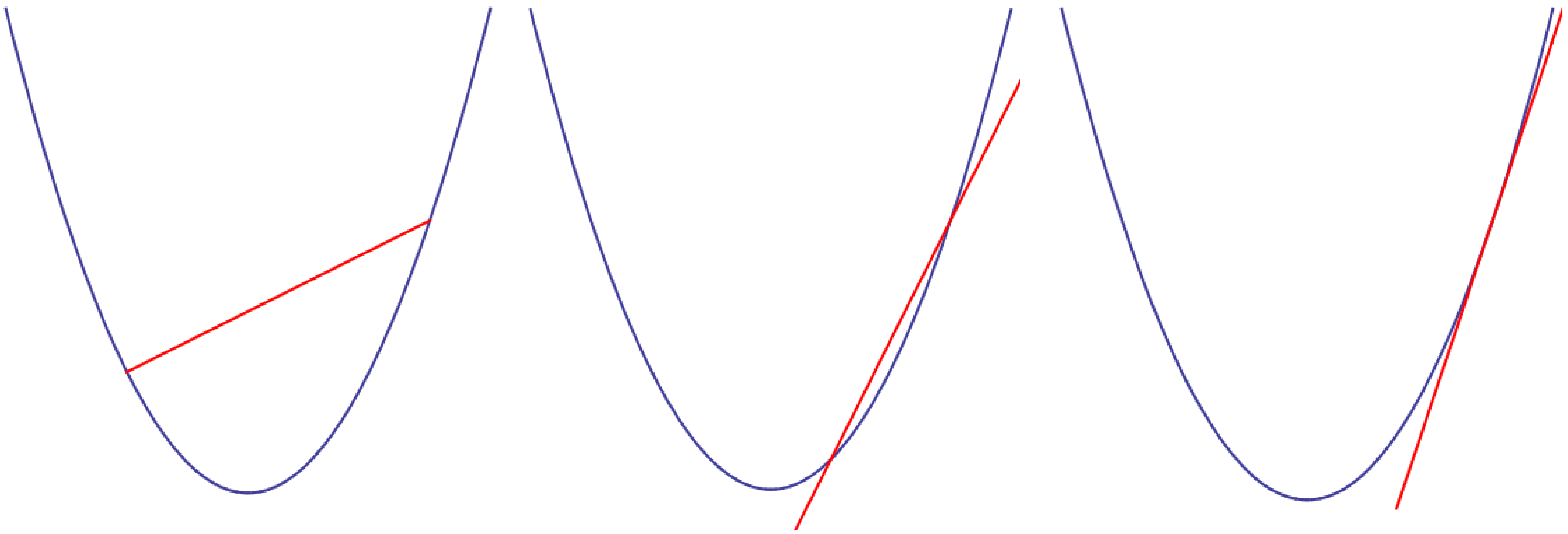


$$-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \quad vs. \quad -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$$

What is this material?

# Linear Regime vs. Strict Limit

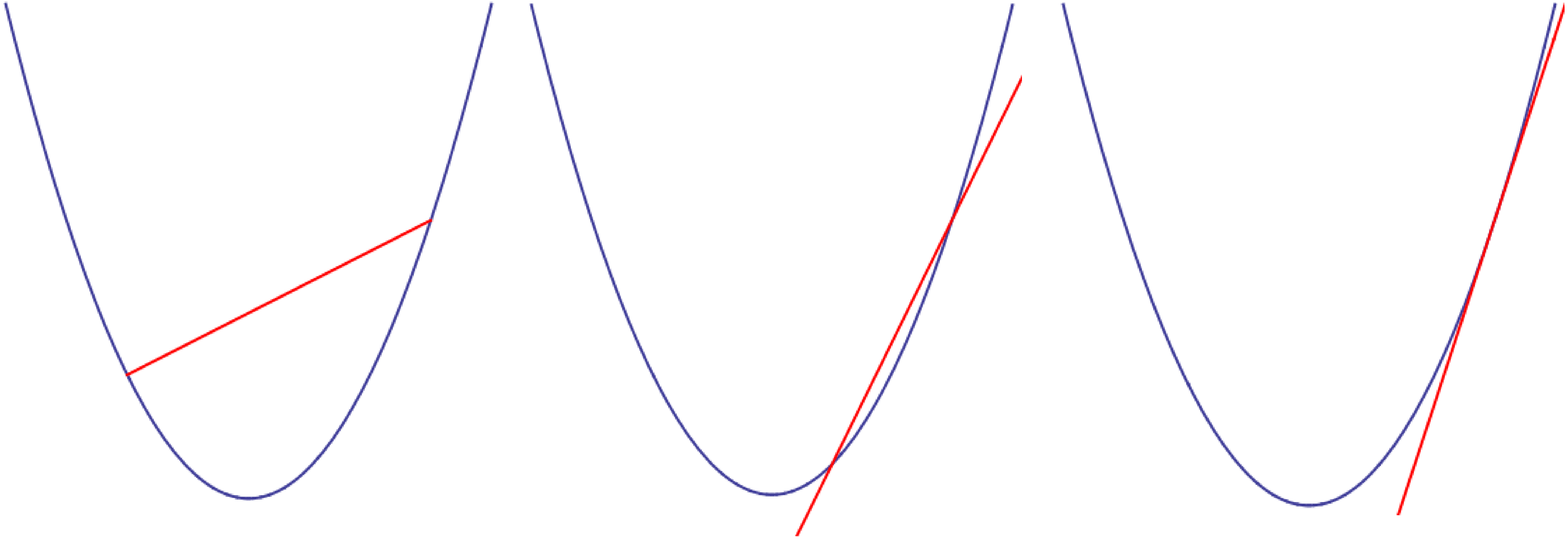
- Which diagram(s) represent the derivative?



- average vs. approximation vs. exact

# Thick Derivatives

- What counts as a derivative?
  - Mathematicians: bright line at strict derivative.
  - Physicists: bright line at “good enough.”



# Notations for Partial Derivatives

- Math vs Physics

$$f_x \equiv \frac{\partial f}{\partial x}$$

- Mechanics

$$\vec{f} = f_x \hat{x} + f_y \hat{y}$$

- E & M

$$E_x = -\left(\frac{\partial V}{\partial x}\right)$$

# Equations Encode Meaning

$$grad\ f = \langle f_x, f_y, f_z \rangle$$

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

# Which Aspects of Concept Image Are Cued?

- The importance of representations:

  Different representations cue different aspects of a student's concept image.

- Rule of Four:
  - Graphs
  - Equations
  - Words
  - Numerical

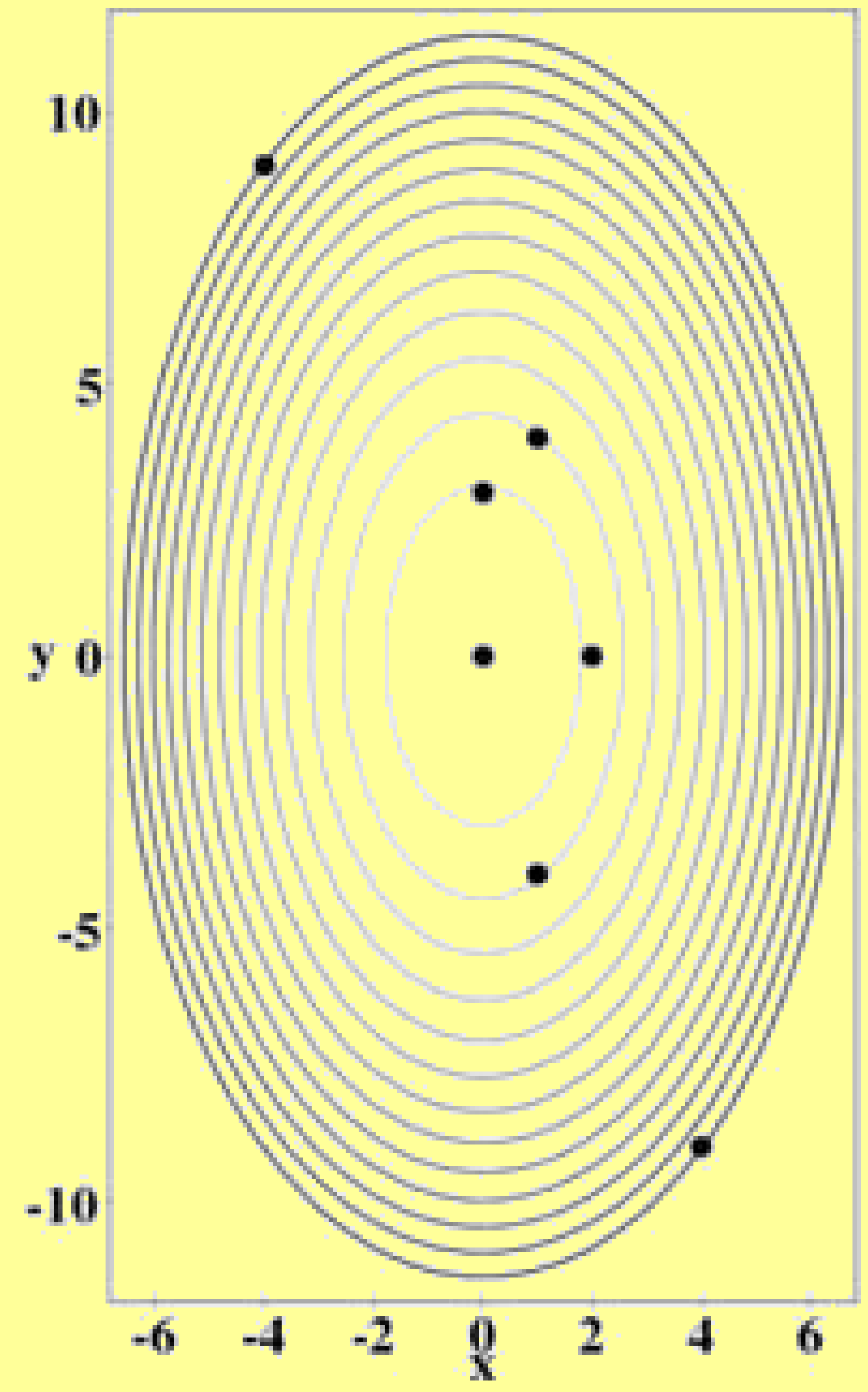
# Concept Image of Gradient

- Use SWBQs to help students link elements of their concept image:

On your small white board, write ONE element of your concept image of gradient.

# Kinesthetic Activity: Gradient

- Points in the direction of steepest change.
- Magnitude is slope.



# Gradient:  Which Direction?

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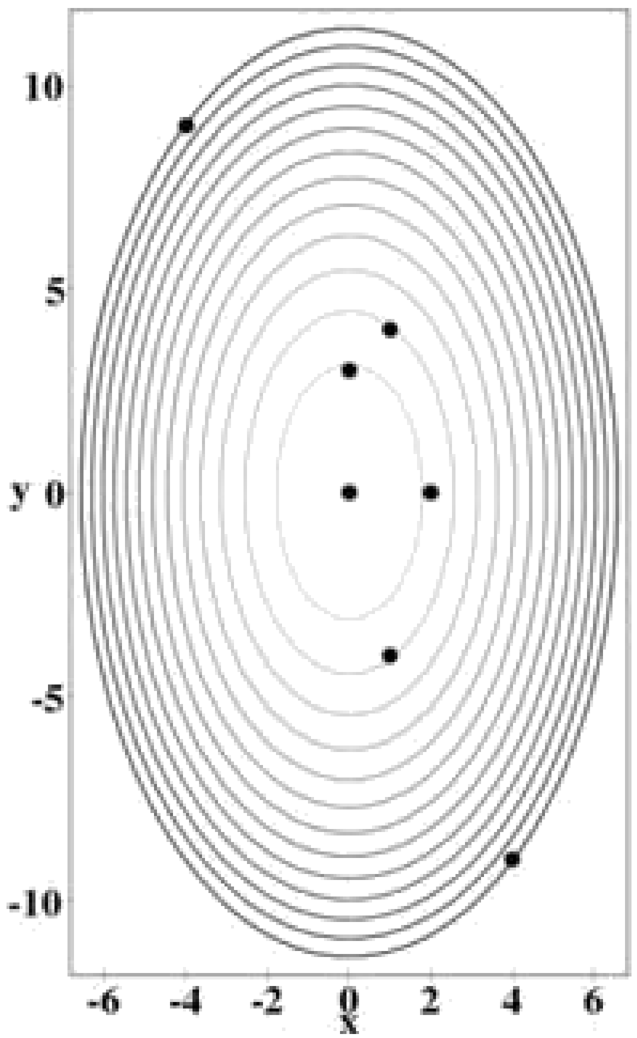
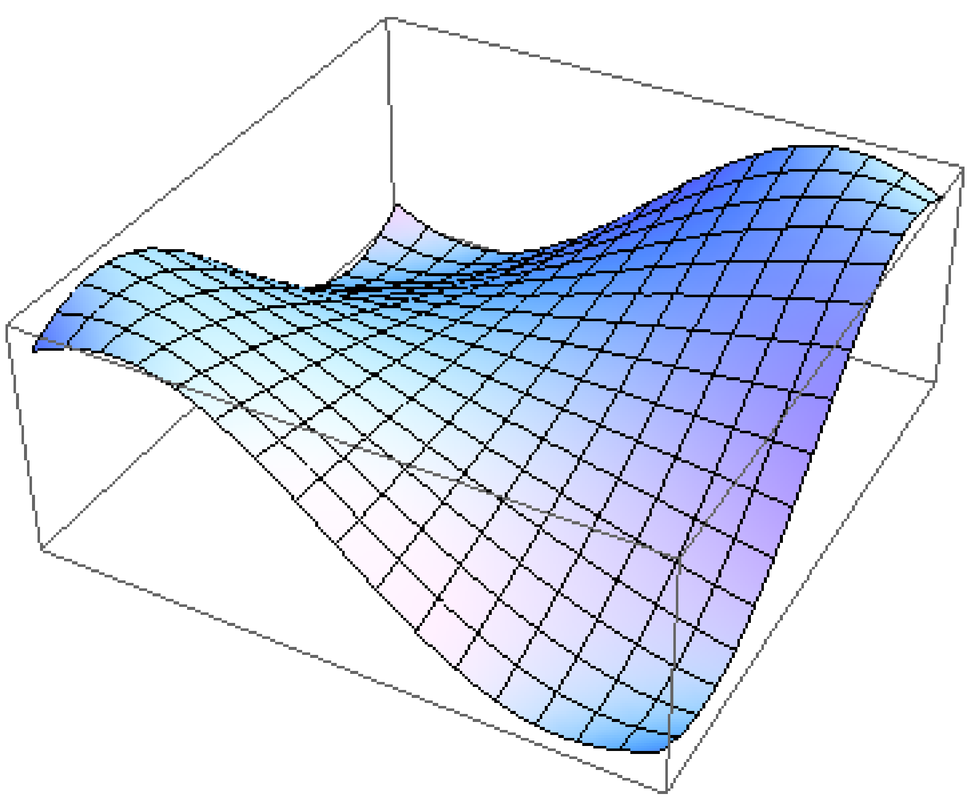
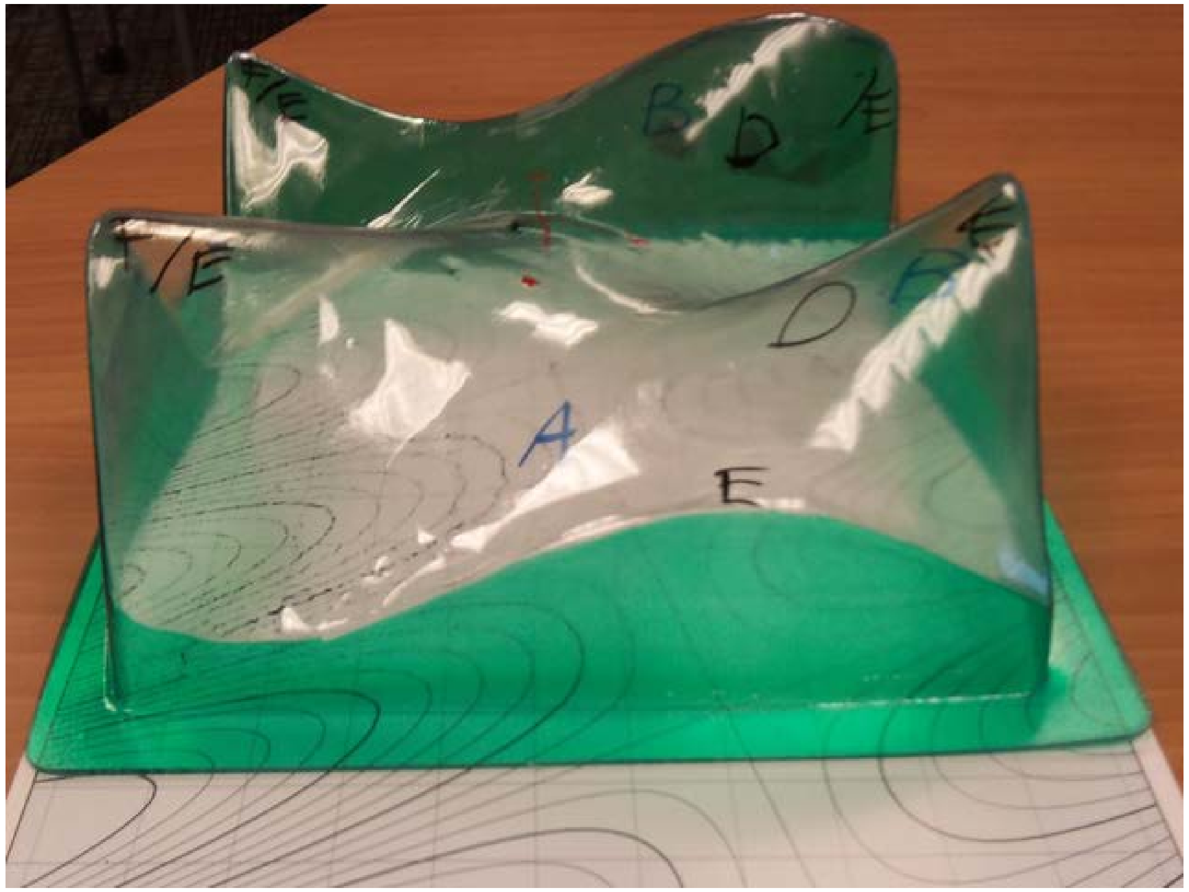
# Math Representations



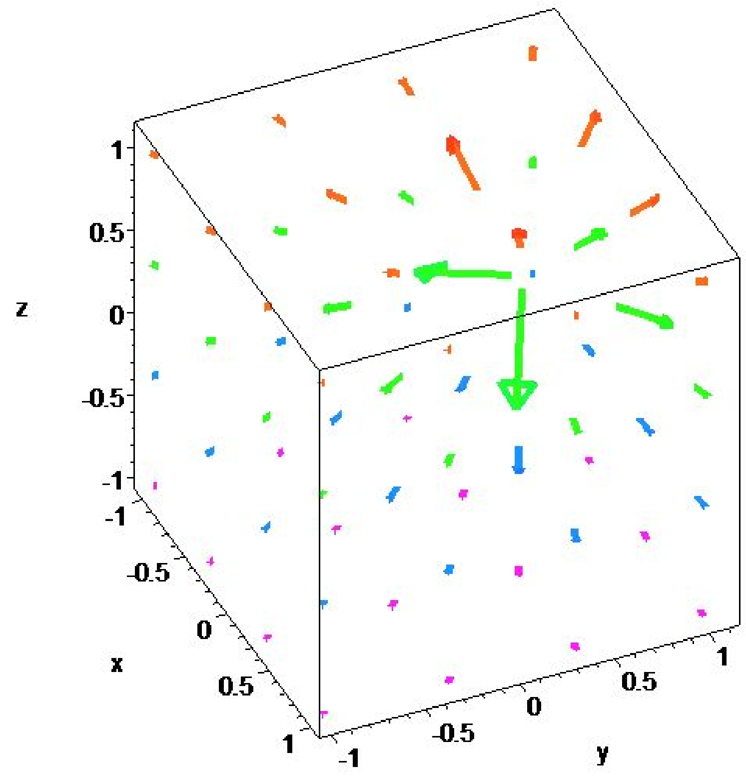

- Functions of 2 variables

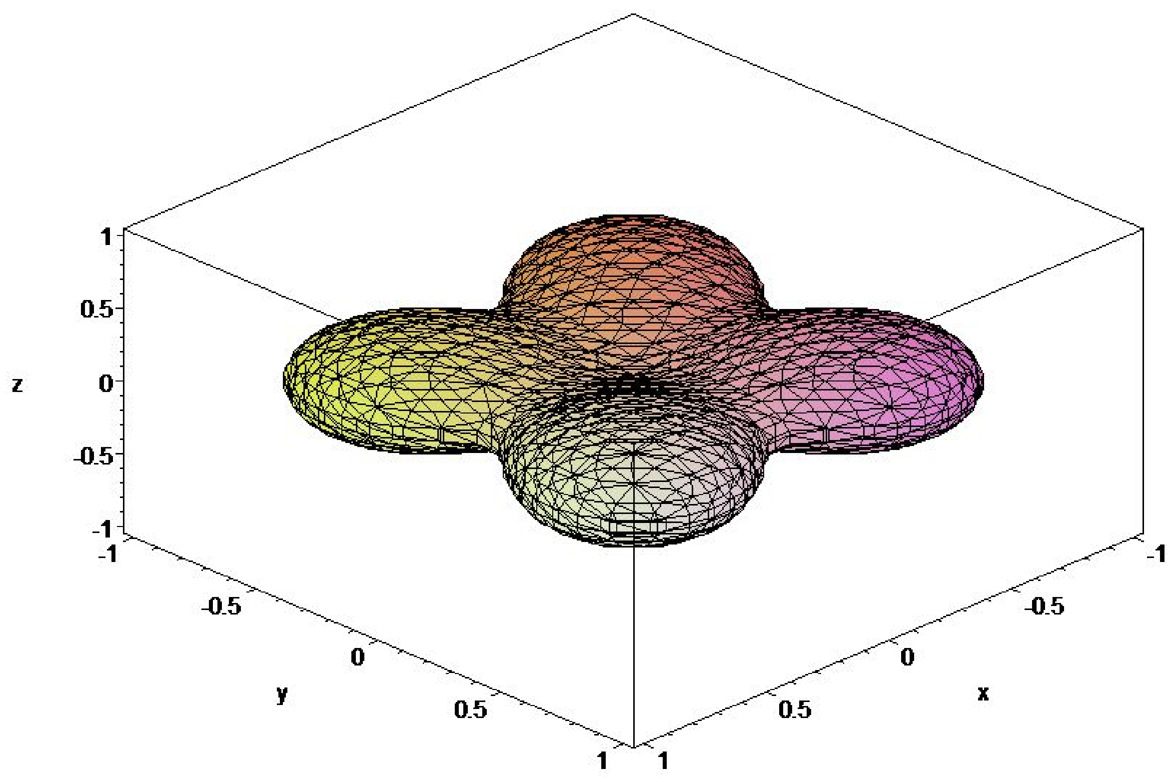


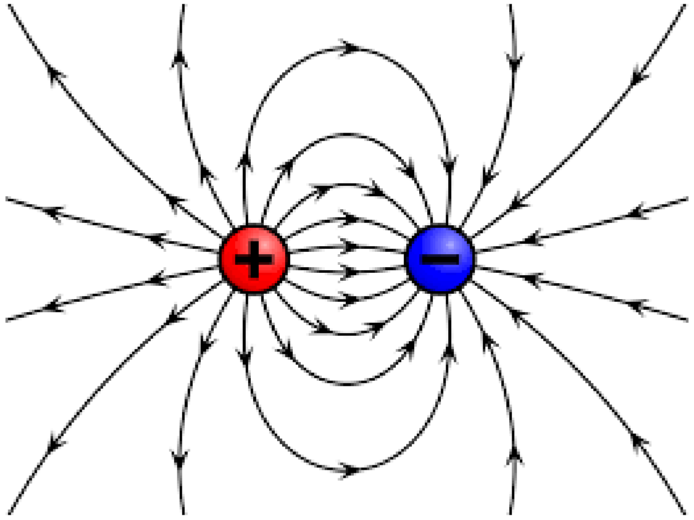


# Physics Representations

- Functions of 3 variables
  - Equipotential Surfaces
  - 3-D Gradient Vectors
  - Electric Field Lines



# Research on Partial Derivatives

- What information can be easily extracted from particular representations?
- How do students change from one representations to another?
- What does expert problem solving look like?

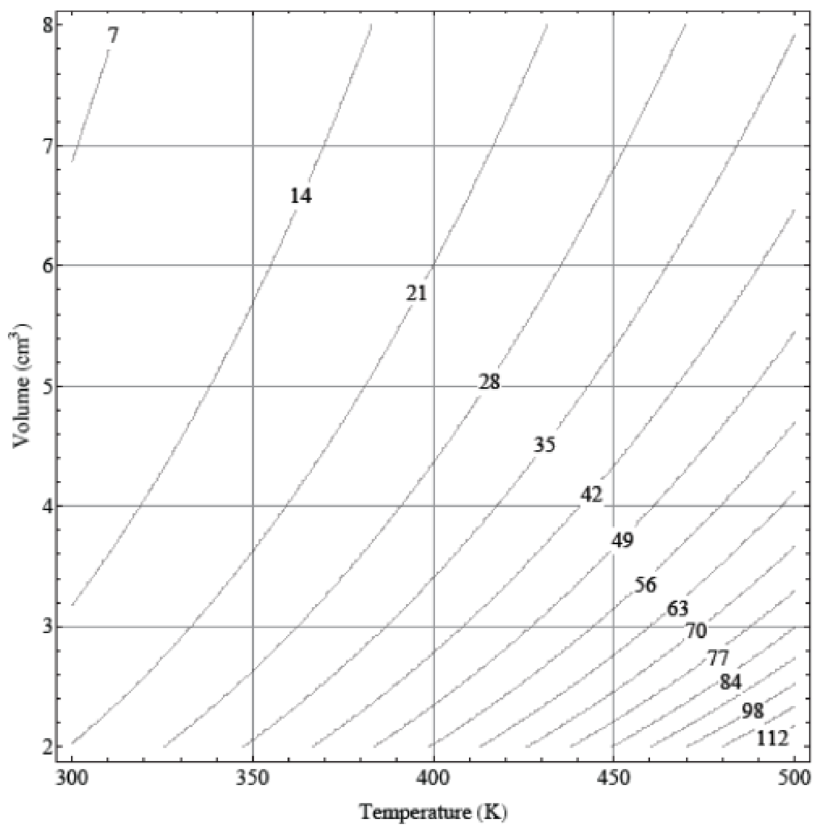
# Representational Transformation

Rabindra Bajracharya

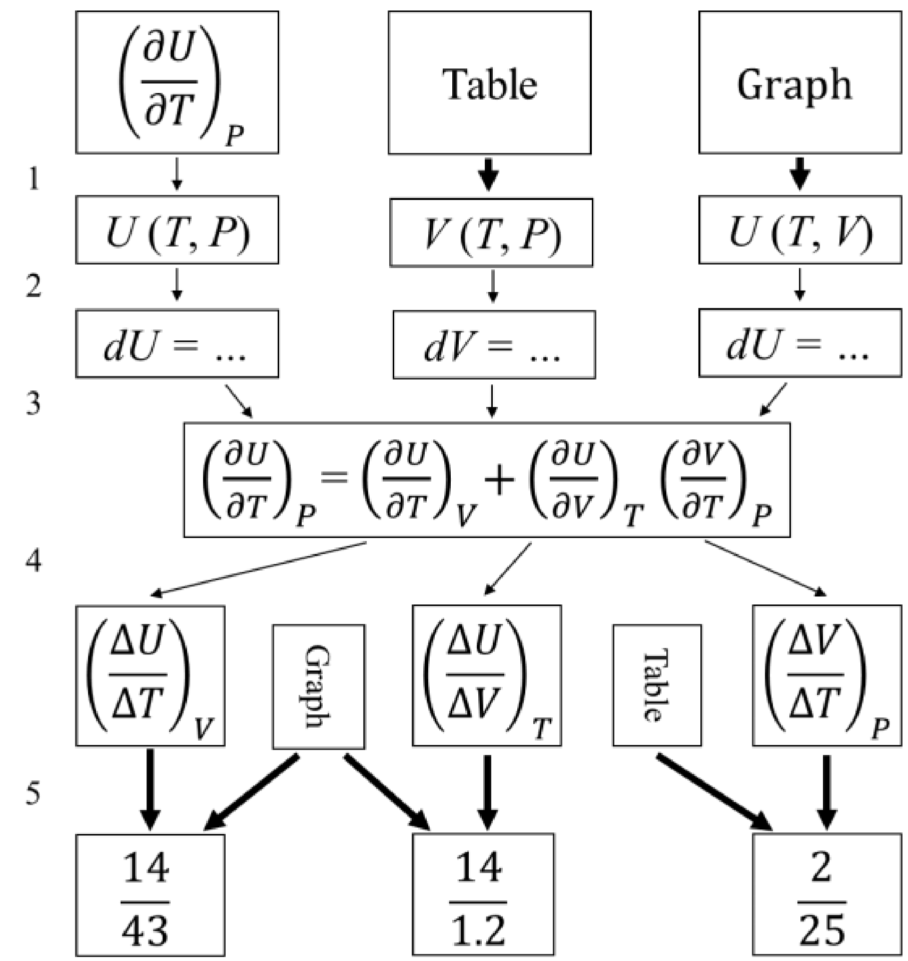
**Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at $P$ = 10 atm., $T$ = 410K using the information below.**

| $P$(atm.) | $T$(K) | $V$(cm$^3$) |
|---|---|---|
| 10 | 300 | 1.32 |
| 10 | 310 | 1.44 |
| 10 | 320 | 1.57 |
| 10 | 330 | 1.71 |
| 10 | 340 | 1.85 |
| 10 | 350 | 2.00 |
| 10 | 360 | 2.15 |
| 10 | 370 | 2.32 |
| 10 | 380 | 2.49 |
| 10 | 390 | 2.67 |
| 10 | 400 | 2.86 |
| 10 | 410 | 3.05 |
| 10 | 420 | 3.25 |
| 10 | 430 | 3.47 |
| 10 | 440 | 3.69 |
| 10 | 450 | 3.91 |
| 10 | 460 | 4.15 |
| 10 | 470 | 4.40 |

Pressure $P$, Temperature $T$, and Volume

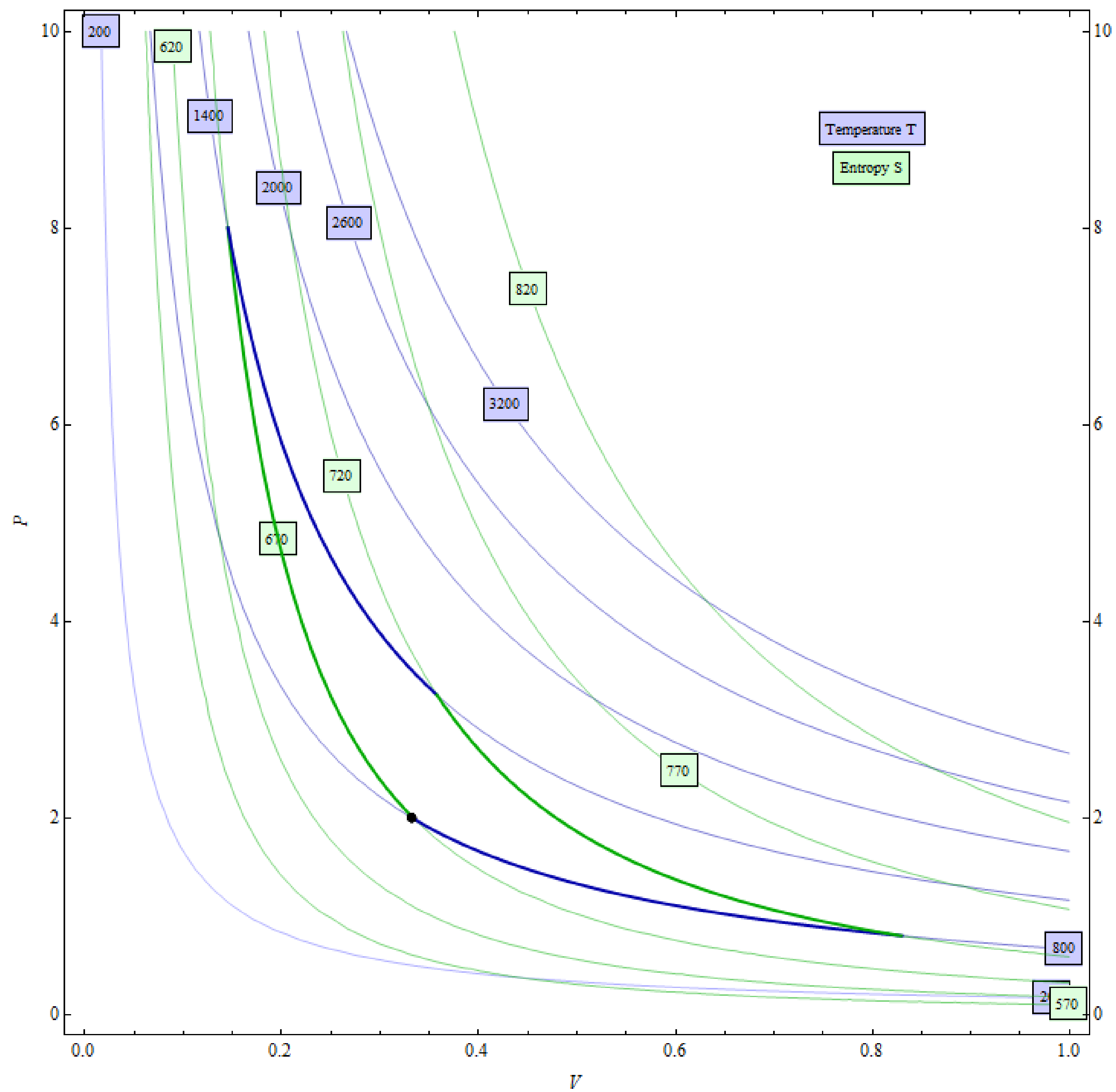



Internal Energy $U$ ($T$, $V$).



Paul Emigh

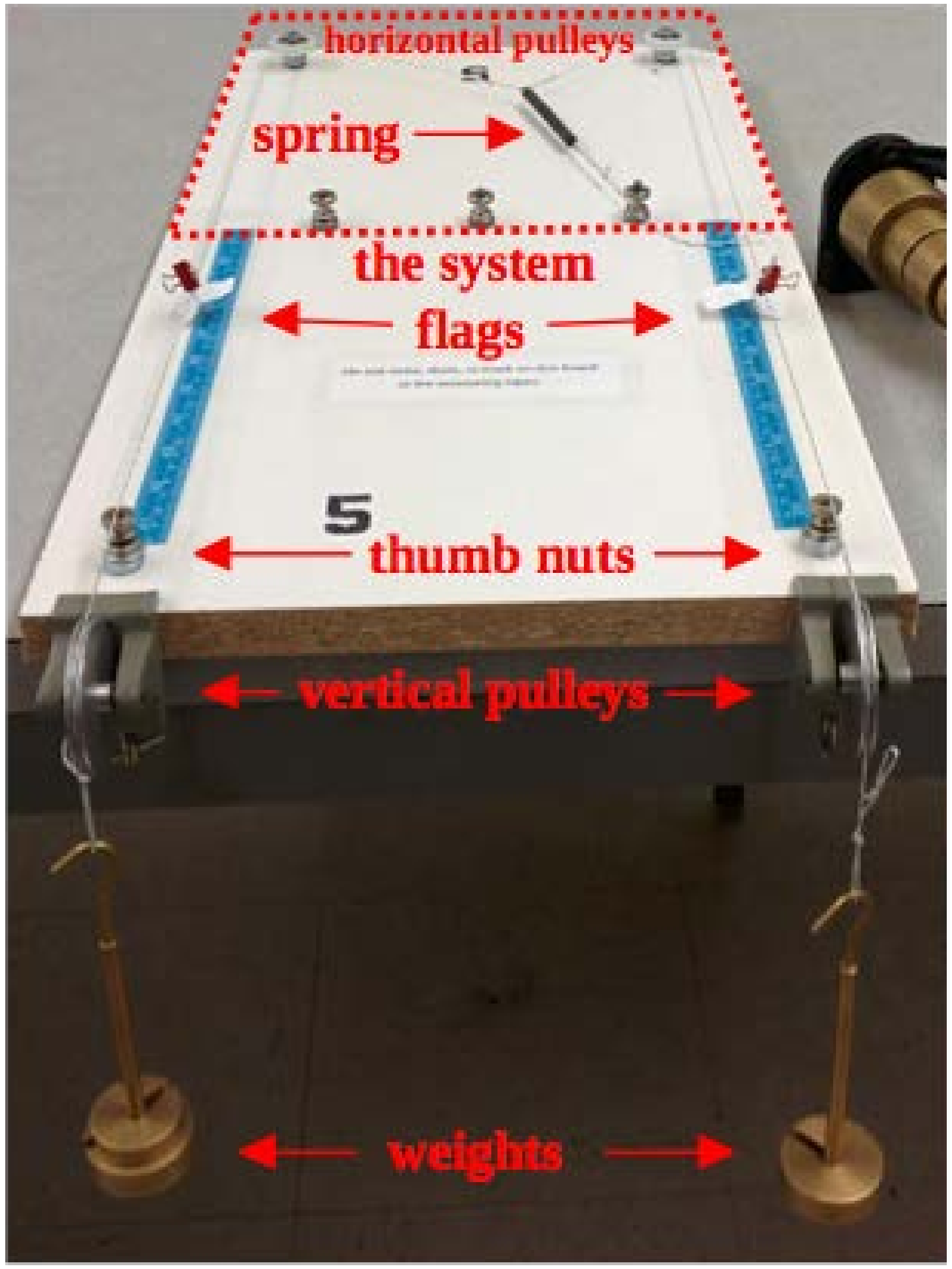
# Contour Maps



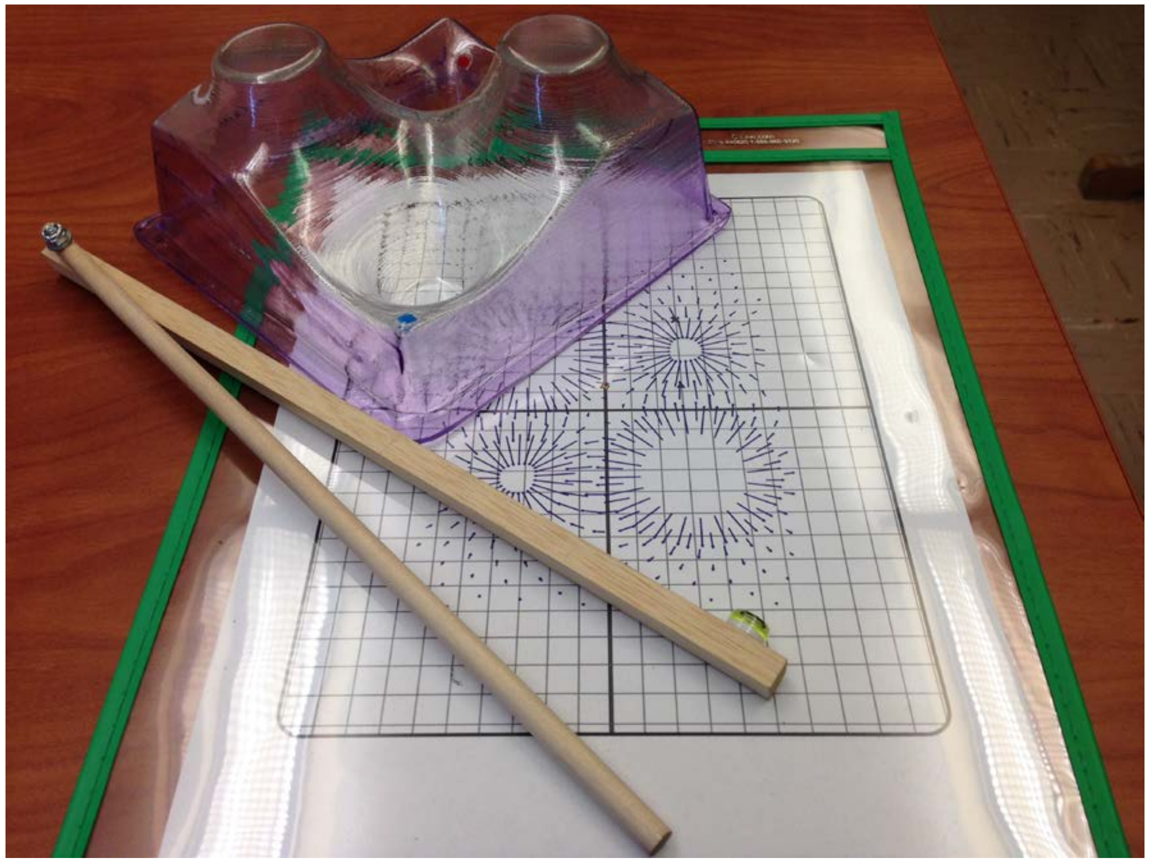
# Partial Derivatives Machine

David Roundy

Mike Vignal

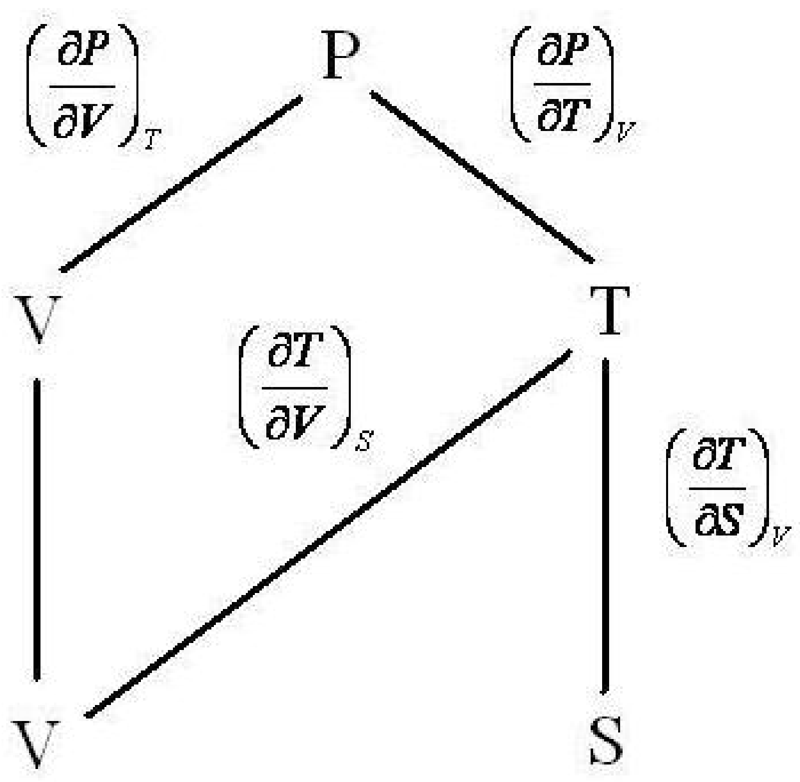


# “Raising” Surfaces/Contour Mats



Elizabeth Gire
Aaron Wangberg
Robyn Wangberg

# Differentials/ Chain Rule Diagrams




Ian Founds

# Experts in Thermo

Mary Bridget Kustusch

$$p(V,T)$$

$$U(V,T)$$

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial V}\right)_S \left(\frac{\partial V}{\partial p}\right)_S \qquad (18)$$

$$S\ (\text{constant}) \rightarrow (V-Nb)T^{3/2}\ (\text{constant}) = C \qquad (19)$$

$$T^{3/2} = C/(V-Nb) \qquad (20)$$

$$T = \left(C/(V-Nb)\right)^{2/3}$$

$$P = \frac{Nk(C)^{2/3}}{(V-Nb)^{5/3}} - \frac{aN^2}{V^2} ; \left[\frac{-5}{3}\frac{NkC^{2/3}}{(V-Nb)^{8/3}} + \frac{2aN^2}{V^3}\right] = \left(\frac{\partial P}{\partial V}\right)_S = \alpha \qquad (21)$$

$$U = \frac{3}{2}Nk\left(\frac{C}{V-Nb}\right)^{2/3} - \frac{aN^2}{V} ; \left[\left(\frac{-2}{3}\right)\frac{3}{2}NkC^{2/3}\left(\frac{1}{V-Nb}\right)^{5/3} + \frac{aN^2}{V^2}\right] = \left(\frac{\partial U}{\partial V}\right)_S = B \qquad (22)$$

$$\left(\frac{\partial U}{\partial P}\right)_S = \left(\frac{\partial U}{\partial V}\right)_S \left(\frac{\partial V}{\partial P}\right)_S = \frac{B}{\alpha}$$

# Conclusion

- The concept image of partial derivative has MANY, many, *many* elements!
- Experts use MANY representations.
- Different representations cue reasoning about different elements.
- Different subfields of mathematics and physics rely on different elements.
- Choose activities that foster connections between elements.
- Learning Progression:  Order matters.