# CLASSIFYING SPACETIMES IN THEORY AND PRACTICE

**Tevian Dray**
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I: Geometry

II: Spacetimes

III: Theory

IV: Practice

# WHICH GEOMETRY?

**flat**

Euclidean

$$ds^2 = dx^2 + dy^2$$

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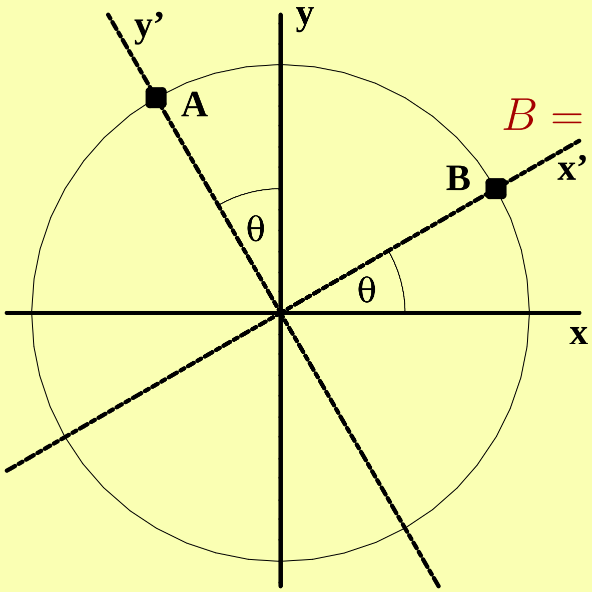


$B = (\cos\theta, \sin\theta)$

*Trigonometry!*

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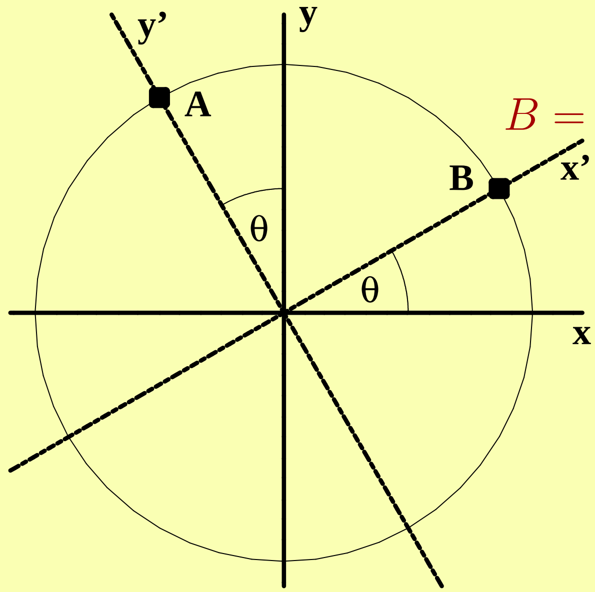


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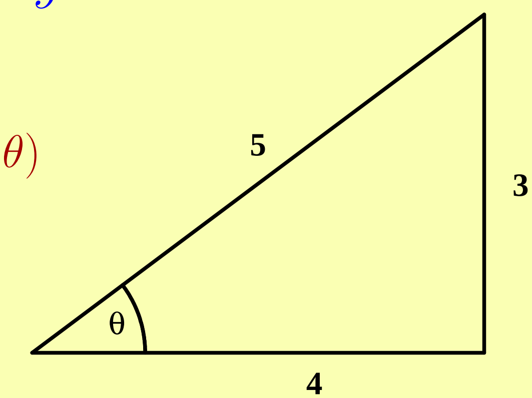
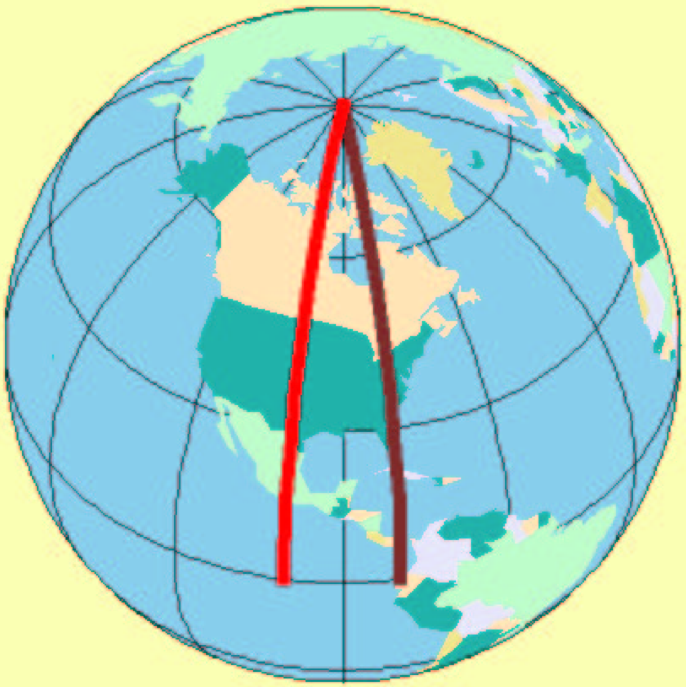



$$\tan\theta = \frac{3}{4}$$

*Trigonometry!*

# WHICH GEOMETRY?

| flat | curved |
|---|---|
| Euclidean | Riemannian |



$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

*Tidal forces!*

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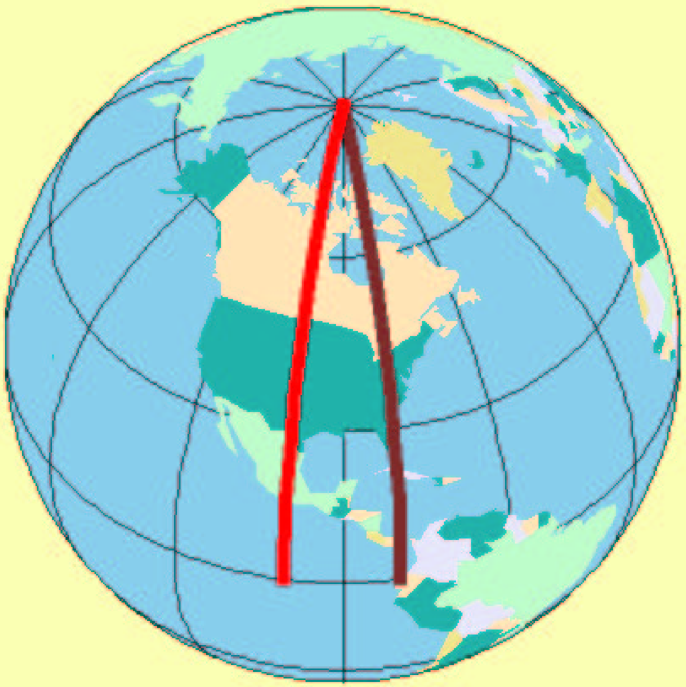

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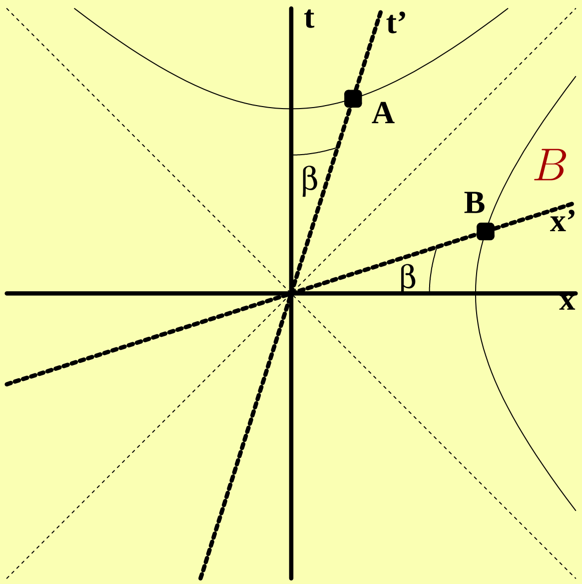


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*Special Relativity!*

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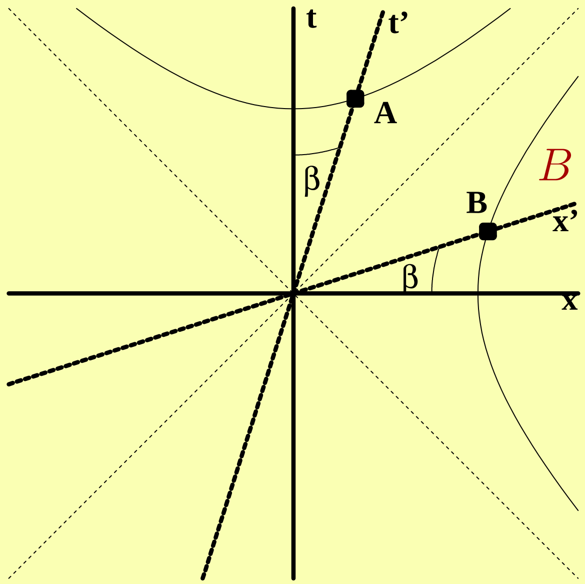


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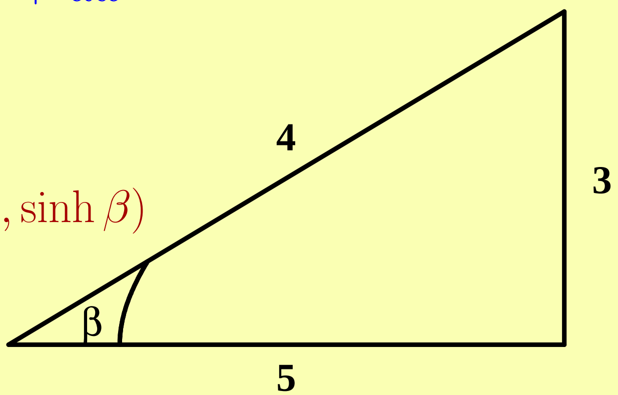


$\tanh\beta = \frac{3}{5}$

*Special Relativity!*

# WHICH GEOMETRY?

| **signature** | **flat** | **curved** |
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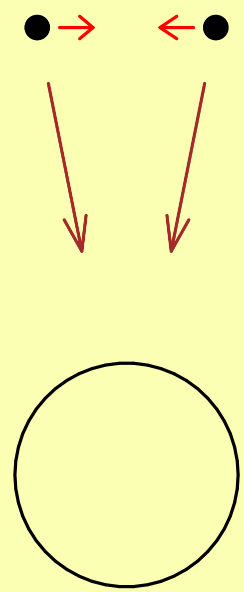
$$ds^2 = -dt^2 + a(t)\, dx^2$$

*Cosmology!*



$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

*General Relativity!*

# CLASSIFYING SPACETIMES

$$
\begin{aligned}
dx^2 + dy^2 &= dr^2 + r^2\,d\phi^2 \\
&= e^{2\rho}\left(d\rho^2 + d\phi^2\right) \\
&= (u^2 + v^2)\left(du^2 + dv^2\right) \\
&\neq r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \\
&= \frac{4e^{2\psi}r^2}{(1+e^{2\psi})^2}\left(d\psi^2 + d\phi^2\right)
\end{aligned}
$$

$$
(\rho = \ln r; \quad x = \tfrac{u^2-v^2}{2},\ y = uv; \quad \psi = \ln\tan\tfrac{\theta}{2})
$$

**When are 2 metrics "the same"?**

## IDEA

**Calculate invariants like curvature!**

**plane:**

$$R = 0$$

**sphere:**

$$R = \frac{2}{r^2}$$

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**Calculate invariants like curvature!**

**plane:**

$$R = 0$$

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**Not the whole story...**

4-d $\Longrightarrow$ 14 scalar algebraic curvature invariants

$\exists$ plane wave solutions where all invariants vanish

# THEORY

$$
\begin{aligned}
ds^2 &= g_{ij}\, dx^i\, dx^j \\
(\, g^{ij}\,) &= (\, g_{ij}\,)^{-1} \\
2\,\Gamma^k{}_{ij} &= g^{km} \left( \frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) \\
R^m{}_{ijk} &= \frac{\partial \Gamma^m{}_{ik}}{\partial x^j} - \frac{\partial \Gamma^m{}_{ij}}{\partial x^k} + \Gamma^m{}_{nj}\Gamma^n{}_{ik} - \Gamma^m{}_{nk}\Gamma^n{}_{ij} \\
R_{ij} &= R^m{}_{imj} \\
R &= g^{ij} R_{ij} \\
G_{ij} &= R_{ij} - \frac{1}{2}\, g_{ij} R
\end{aligned}
$$

$$\boxed{G = 8\pi T}$$

*Einstein's equation!*

# THEORY

**Christoffel:**

- Calculate $R^i{}_{jkl}$ and derivatives
- Compare the components

$\longmapsto$ *algebraic* equations (no derivatives)

(consistency $\Longleftrightarrow$ $\exists$ transformation)

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## Christoffel:

- Calculate ${R^i}_{jkl}$ and derivatives
- Compare the components

$\longmapsto$ *algebraic* equations (no derivatives)

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## Problem:

4-d $\Longrightarrow$ 20 derivatives; number of components is:

$$29,320,310,074,020$$

**PRACTICE**

## Symmetry:

$79,310$ independent components

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**Karlhede:** *Spinors* (reduces number of derivatives to 7)

$$3156 \quad \text{independent components}$$

# PRACTICE

**Symmetry:**

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**Karlhede:** *Spinors* (reduces number of derivatives to 7)

$3156$ independent components

No known example uses more than 3 derivatives

$430$ independent components

# COMPUTER ALGEBRA

- **1960s:** Ray d'Inverno writes LAM (LISP Algebraic Manipulator)
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**Bondi metric:**

$$ds^2 = -\left(\frac{V}{r} - U^2 r^2 e^{2\gamma}\right) du^2 - 2e^{2\beta}\,dr\,du - 2Ur^2 e^{2\gamma}\,du\,d\theta + r^2\left(e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin^2\theta\,d\phi^2\right)$$

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| | |
|---|---|
| Bondi: | 6 months |
| LAM: | 4 minutes (and found 6 errors!) |
| SHEEP/CLASSI: | $<$ 1 second |

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| mixed | Signature Change! | |