The Geometry of General Relativity

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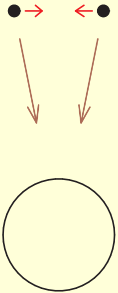
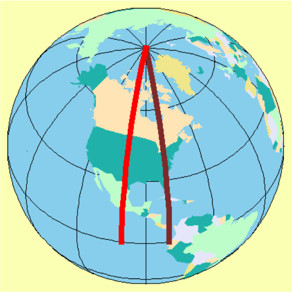
Which Geometry?

	$s = 0$	$s = 1$
flat	Euclidean	Minkowskian **(SR)**
curved	Riemannian	Lorentzian **(GR)**



Tidal forces!

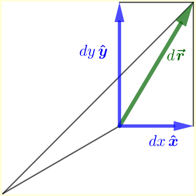


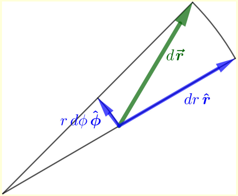
Need calculus to describe curvature!

Vector Calculus in a Nutshell

Vector calculus is about one coherent concept:

Infinitesimal Displacement





$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$$

$$ds = |d\vec{r}|$$

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$$

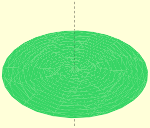
$$dA = |d\vec{r}_1 \times d\vec{r}_2|$$

$$dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3$$

$$df = \vec{\nabla} f \cdot d\vec{r}$$

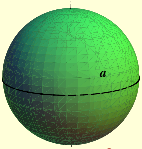
Line Elements

$$ds^2 = d\vec{r} \cdot d\vec{r}$$



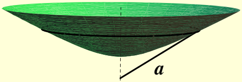
$dr^2 + r^2\,d\phi^2$

plane



$d\theta^2 + \sin^2\theta\,d\phi^2$

sphere



$d\beta^2 + \sinh^2\beta\,d\phi^2$

hyperboloid

Euclidean Geometry: $ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$

Special Relativity: $ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$

Black Hole: $ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$

Cosmology: $ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right)$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

Differential forms are integrands: ($*^2 = 1$)

$$
\begin{aligned}
f &= f && \text{(0-form)}\\
F &= \vec{F} \cdot d\vec{r} && \text{(1-form)}\\
*F &= \vec{F} \cdot d\vec{A} && \text{(2-form)}\\
*f &= f\, dV && \text{(3-form)}
\end{aligned}
$$

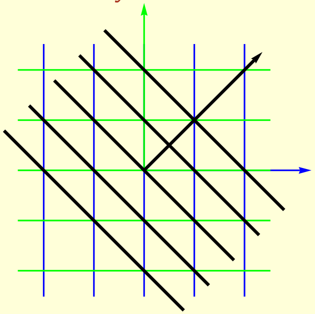
Products:

$$
\begin{aligned}
F \wedge G &= \vec{F} \times \vec{G} \cdot d\vec{A}\\
F \wedge *G &= \vec{F} \cdot \vec{G}\, dV
\end{aligned}
$$

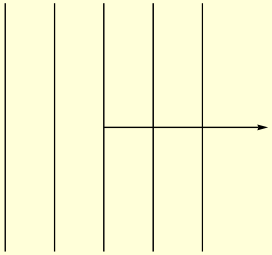
Exterior derivative: ($d^2 = 0$)

$$
\begin{aligned}
df &= \vec{\nabla} f \cdot d\vec{r}\\
dF &= \vec{\nabla} \times \vec{F} \cdot d\vec{A}\\
d{*F} &= \vec{\nabla} \cdot \vec{F}\, dV\\
d{*f} &= 0
\end{aligned}
$$

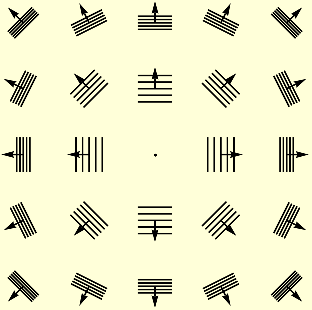
The Geometry of Differential Forms



$dx + dy$



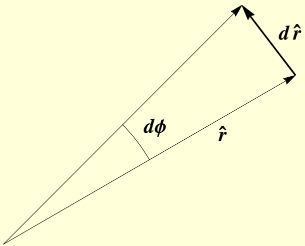
dx

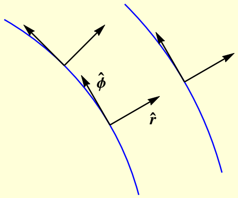


$r\,dr = x\,dx + y\,dy$

Derivatives of Vectors

Rectangular basis vectors are constant: $d\hat{x} = 0$, $d\hat{y} = 0$
What about polar basis vectors?





$$d\hat{r} = d\phi\,\hat{\phi}$$

$$d\hat{\phi} = -d\phi\,\hat{r}$$

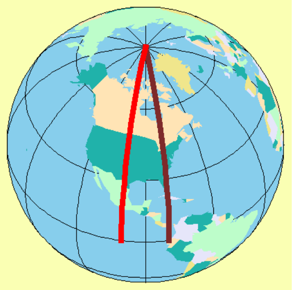
Geodesics are curves that are as straight as possible.

Flat space (Euclidean or Minkowskian):

Geodesics are straight lines.

Sphere:

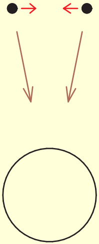
Geodesics are great circles.



General Relativity:

Freely falling objects move along geodesics.

(future-pointing, timelike)



Geodesic Equation

Orthonormal basis: $$\boxed{d\vec{r} = \sigma^i\,\hat{e}_i}$$

Connection:

$$d\hat{e}_j = \omega^i{}_j\,\hat{e}_i$$

$$d^2\vec{r} = 0 \Longrightarrow d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$d(\vec{u}\cdot\vec{w}) = d\vec{u}\cdot\vec{w} + \vec{u}\cdot d\vec{w} \Longrightarrow \omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{v}\,d\lambda = d\vec{r} \qquad (\vec{v} = \dot{\vec{r}})$$

$$d\vec{v} = 0 \qquad (\dot{\vec{v}} = 0)$$

Curvature: $$d^2\hat{e}_j = \Omega^i{}_j\,\hat{e}_i \Longrightarrow \Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein's Equation

Curvature: $d^2\hat{e}_j = \Omega^i{}_j \hat{e}_i \Longrightarrow \Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{G} = G^i \hat{e}_i = G^i{}_j\,\sigma^j \hat{e}_i$$

$$\Longrightarrow d{*}\vec{G} = 0 \qquad (\vec{\nabla} \cdot \vec{G} = 0)$$

Field equation:

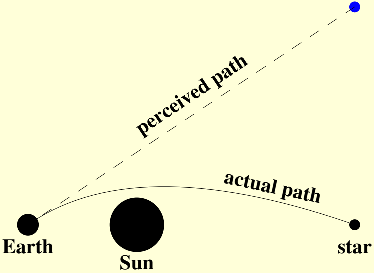
$$\boxed{\vec{G} + \Lambda\, d\vec{r} = 8\pi \vec{T}}$$

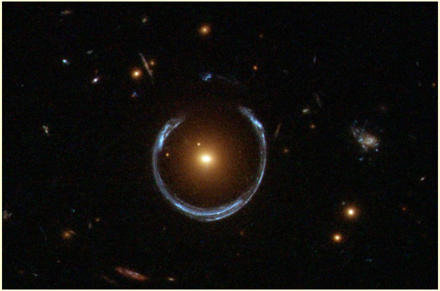
(curvature = matter)

[10 coupled, 2nd-order, nonlinear PDEs ...]

$\Longrightarrow \vec{\nabla} \cdot \vec{T} = 0$: Energy/Momentum Conservation!

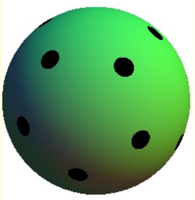
Gravitational Lensing





https://commons.wikimedia.org/wiki/File:Lensshoe_hubble.jpg

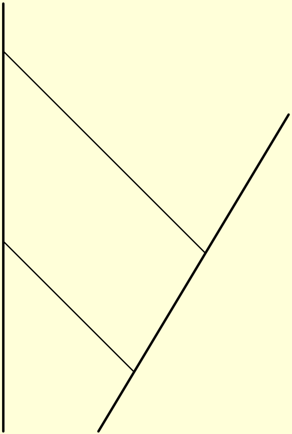
Cosmological Redshift



$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right)$$

$$\boxed{1 + z = \frac{a(t_R)}{a(t_E)} \approx 1 + \frac{\dot{a}}{a} \Delta s}$$

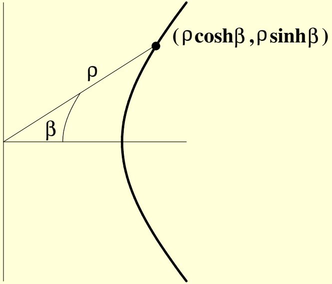
(redshift $\propto$ distance)

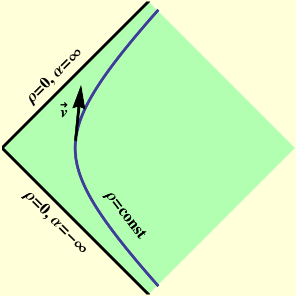


(Same idea as Doppler shift!)

Rindler Geometry

constant curvature = constant acceleration

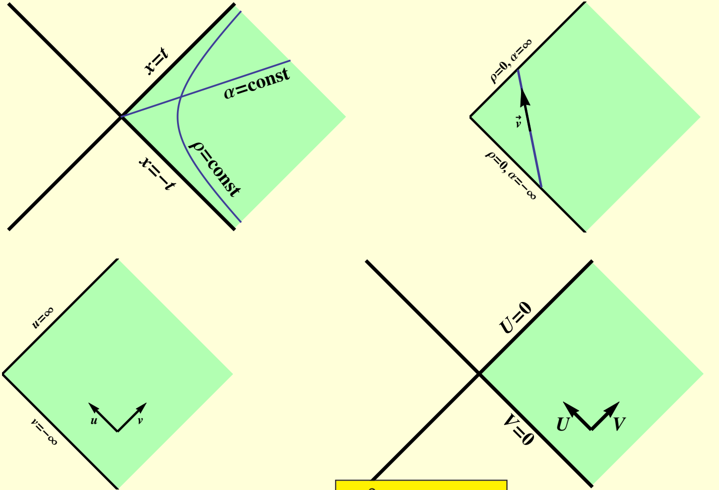




Can outrun lightbeam!

$$x = \rho \cosh \alpha \qquad t = \rho \sinh \alpha \quad \Longrightarrow \quad ds^2 = d\rho^2 - \rho^2\, d\alpha^2$$

From Rindler to Minkowski

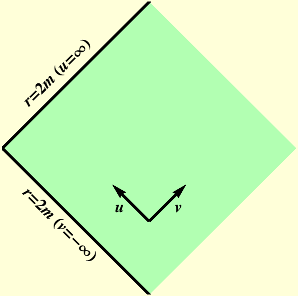


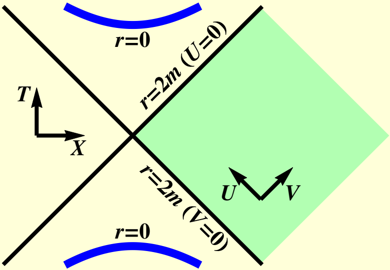
$\boxed{ds^2 = -dU\,dV} = -\mathrm{d}(\mathrm{t}-\mathrm{x})\,\mathrm{d}(\mathrm{t}+\mathrm{x})$

$u = \alpha - \ln\rho,\ v = \alpha + \ln\rho \qquad U = -e^{-u} = -\rho\, e^{-\alpha},\ V = e^{v} = \rho\, e^{\alpha}$

From Schwarzschild to Kruskal

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\, dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right)$$

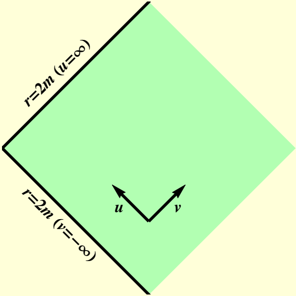


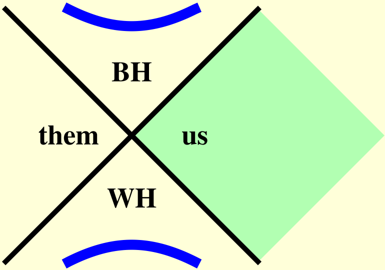


$$ds^2 = -\frac{32m^3}{r} e^{-r/2m}\, dU\, dV$$

From Schwarzschild to Kruskal

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\, dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

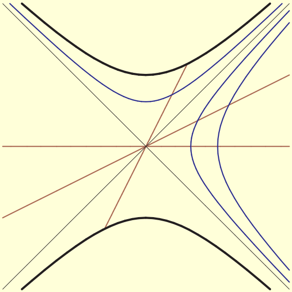




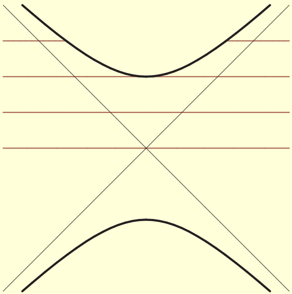
$$ds^2 = -\frac{32m^3}{r} e^{-r/2m}\, dU\, dV$$

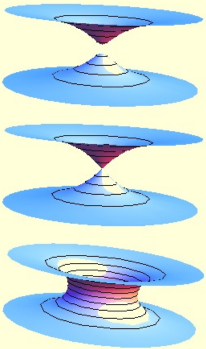
Wormholes

Constant radius = constant acceleration!

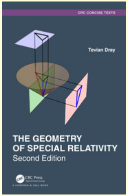


Wormholes

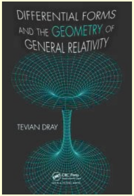




SUMMARY



http://relativity.geometryof.org/GSR
http://relativity.geometryof.org/GDF
http://relativity.geometryof.org/GGR



- Special relativity = hyperbolic trigonometry
- General relativity = Lorentzian vector calculus
- Curvature = gravity
- Geometry = physics

THE END