# The OCTONIONIC EIGENVALUE Problem

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I: Eigenvalues
II: Quaternions
III: Octonions
IV: Octonionic Eigenvectors
V: Applications

# DIVISION ALGEBRAS

**Real Numbers:**

$$\mathbb{R}$$

**Complex Numbers:**

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$

$$z = x + yi$$

$$\boxed{i^2 = -1}$$

# COMPLEX EIGENVALUE PROBLEM

**Eigenvalue Equation:** $Av = \lambda v$

**Hermitian:** $A^\dagger = \overline{A}^T = A$

**Reality:** $\overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v$

**Orthogonality:** $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

since: $\lambda_1 v_1^\dagger v_2 = (A v_1)^\dagger v_2 = v_1^\dagger A v_2 = v_1^\dagger \lambda_2 v_2$

- $\exists\, n$ eigenvalues, all real.
- $\exists$ basis of orthonormal eigenvectors.

**Decomposition:** $A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$

# QUANTUM MECHANICS

$$v \longleftrightarrow |v\rangle$$
$$v^\dagger \longleftrightarrow \langle v|$$

**Normalization:** $\langle v|v\rangle = 1$

**Measurement:** $A\,|v_m\rangle = \lambda_m\,|v_m\rangle \qquad (\lambda_m \in \mathbb{R}!)$

**Simultaneous Measurement:** $[A, B] = AB - BA = 0$

(same eigenvectors!)

**Decomposition:**
$$A = \sum_{m=1}^{n} \lambda_m\,|v_m\rangle\langle v_m|$$

# DIVISION ALGEBRAS

**Real Numbers:**

$$\mathbb{R}$$

**Quaternions:**

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$
$$q = (a + bi) + (c + di)j$$

**Complex Numbers:**

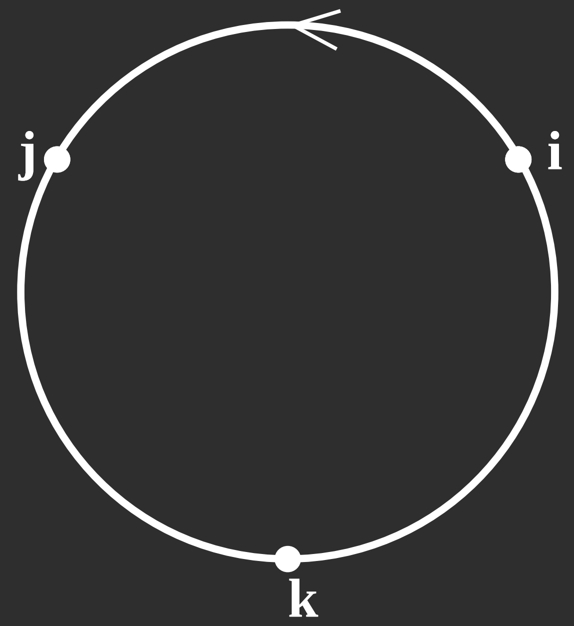
$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$
$$z = x + yi$$

$$i^2 = j^2 = -1$$

# QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

not commutative



# VECTORS I

$$v = bi + cj + dk \longleftrightarrow \vec{\boldsymbol{v}} = b\,\hat{\boldsymbol{\imath}} + c\,\hat{\boldsymbol{\jmath}} + d\,\hat{\boldsymbol{k}}$$

$$vw \longleftrightarrow -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$$

Dot product exists in any dimension
but
Cross product exists only in 3 and 7 dimensions

# THE DISCOVERY OF THE QUATERNIONS

Brougham Bridge (Dublin)


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

# QUATERNIONIC EIGENVALUE PROBLEM

**Eigenvalue Equation:** $Av = \lambda v$

**Hermitian:** $A^\dagger = A$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1-j \\ k-i \end{pmatrix} = (1-j) \begin{pmatrix} 1 \\ k \end{pmatrix}$$

Eigenvalues need not be real!

Furthermore, $A(qv) \neq q(Av)$, so:

Multiples of eigenvectors not eigenvectors!

# RIGHT EIGENVALUE PROBLEM

**Eigenvalue Equation:** $Av = v\lambda$

**Hermitian:** $A^\dagger = A$

**Reality:**

$$\begin{aligned}\overline{\lambda}(v^\dagger v) &= (\overline{\lambda} v^\dagger)v = (Av)^\dagger v = (v^\dagger A)v \\ &= v^\dagger (Av) = v^\dagger (v\lambda) = (v^\dagger v)\lambda\end{aligned}$$

**Multiples:**

$$A(vq) = (Av)q = (v\lambda)q = vq\lambda$$

- $\exists\, n$ eigenvalues, all real.
- $\exists$ basis of orthonormal eigenvectors.

**Decomposition:**

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$$

# DIVISION ALGEBRAS

**Real Numbers:**

$$\mathbb{R}$$

**Quaternions:**

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$
$$q = (a + bi) + (c + di)j$$

**Complex Numbers:**

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$
$$z = x + yi$$

**Octonions:**

$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

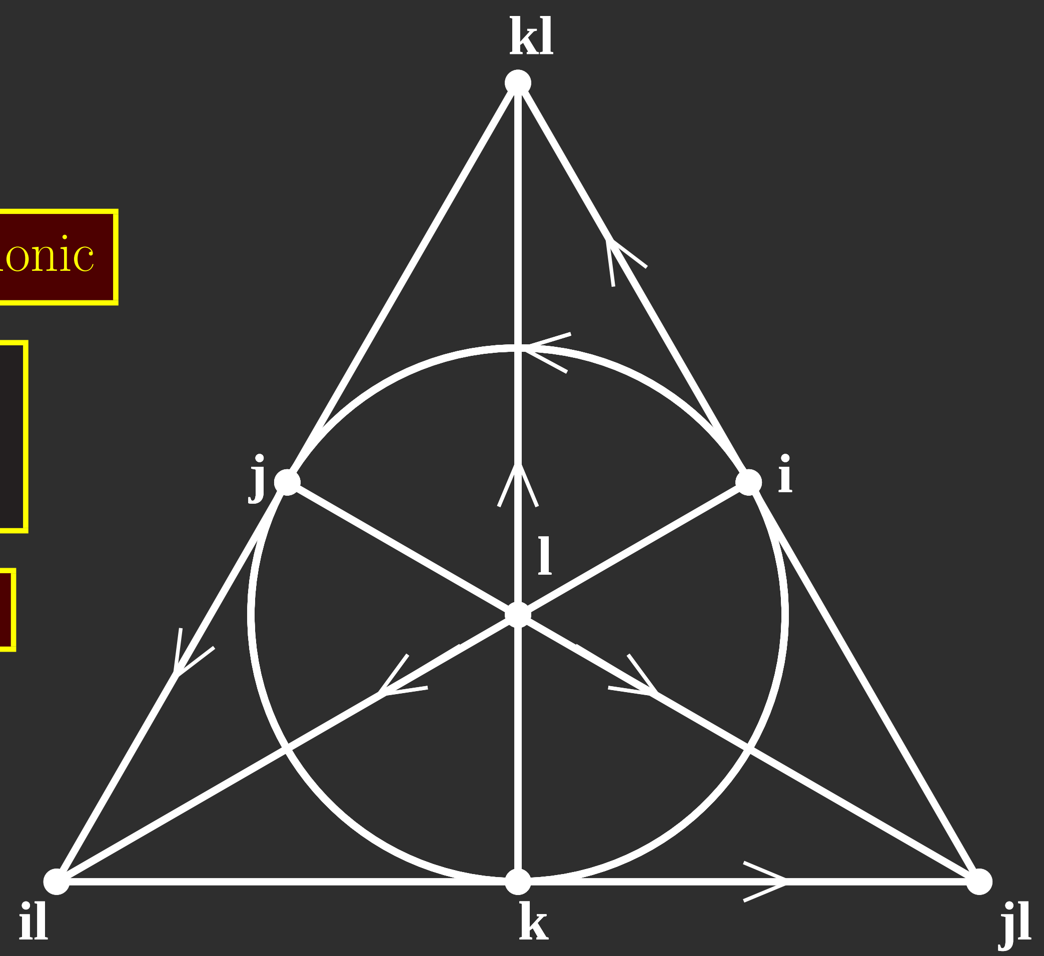
$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

# OCTONIONS

each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative



# WHAT STILL WORKS?

- $q \in \operatorname{Im} \mathbb{O} \Longrightarrow \bar{q} = -q$
- $q\bar{q} = |q|^2$
- $q^{-1} = \dfrac{\bar{q}}{|q|^2} \qquad (q \neq 0)$
- $|pq| = |p||q|$
- $[p, p, q] = (pp)q - p(pq) = 0$

alternativity

# THE DISCOVERY OF THE OCTONIONS

Brougham Bridge (Dublin)


John T. Graves (1843!)
Arthur Cayley (1845)
octaves, Cayley numbers

# OCTONIONIC EIGENVALUE PROBLEM

**Eigenvalue Equation:** $Av = v\lambda$
**Hermitian:** $A^\dagger = A$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} j \\ \ell \end{pmatrix} = \begin{pmatrix} j + i\ell \\ \ell - k \end{pmatrix} = \begin{pmatrix} j \\ \ell \end{pmatrix} (1 - k\ell)$$

Eigenvalues need not be real!

$$\begin{aligned} \overline{\lambda}(v^\dagger v) &\neq (\overline{\lambda} v^\dagger)v = (Av)^\dagger v = (v^\dagger A)v \\ &\neq v^\dagger (Av) = v^\dagger (v\lambda) \neq (v^\dagger v)\lambda \end{aligned}$$

## 3×3 REAL EIGENVALUE PROBLEM

**Characteristic Equation:**

$$\mathcal{A}^3 - (\operatorname{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det\mathcal{A})\,I = 0$$

$$\mathcal{A}^3 = \frac{1}{2}(\mathcal{A}\mathcal{A}^2 + \mathcal{A}^2\mathcal{A})$$

**But:**

$$\lambda^3 - (\operatorname{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - \det\mathcal{A} = r$$

$$(r - r_+)(r - r_-) = 0$$

- 2 sets of 3 real eigenvalues!
- 4-parameter families of eigenvectors
- $\det\mathcal{A} = 0 \quad \not\Longrightarrow \quad \lambda = 0$!

# DECOMPOSITIONS I

**Family:** $K[v] = rv =$

$$\mathcal{A}\Big(\mathcal{A}\big(\mathcal{A}(v)\big)\Big) - (\mathrm{tr}\mathcal{A})\,\mathcal{A}\big(\mathcal{A}(v)\big) + \sigma(\mathcal{A})\,\mathcal{A}(v) - \det(\mathcal{A})\,v$$

*Theorem:* ($v$, $w$ in same family; $\lambda \neq \mu$)

$$(vv^\dagger)w = 0$$

*Proof:* 8 hour brute force Mathematica computation!

(Analytic proof by Okubo!)

*Corollary:*

$$\mathcal{A} = \sum \lambda vv^\dagger$$

## PROJECTIONS

**Idea:** $(uu^{\dagger})\,\boldsymbol{z} \longleftrightarrow u\,(u\cdot\boldsymbol{z})$

*Theorem:* ($\boldsymbol{z}$ *any* vector in same family)

$$(uu^{\dagger})\left((uu^{\dagger})\boldsymbol{z}\right) = (uu^{\dagger})\boldsymbol{z}$$

*Corollary:*

$$(vv^{\dagger})\left((uu^{\dagger})\boldsymbol{z}\right) = 0$$

*Corollary:*

$$\mathcal{A}\left((uu^{\dagger})\boldsymbol{z}\right) = \lambda\left((uu^{\dagger})\boldsymbol{z}\right)$$

# DECOMPOSITIONS II

**Into Families:**

$$\boldsymbol{z} = \frac{K[\boldsymbol{z}] - r_2\,\boldsymbol{z}}{r_1 - r_2} - \frac{K[\boldsymbol{z}] - r_1\,\boldsymbol{z}}{r_1 - r_2} = \boldsymbol{z}_1 + \boldsymbol{z}_2$$

**Within Families:** $(uu^\dagger + vv^\dagger + ww^\dagger = I)$

$$\boldsymbol{z}_1 = (uu^\dagger)\boldsymbol{z}_1 + (vv^\dagger)\boldsymbol{z}_1 + (ww^\dagger)\boldsymbol{z}_1$$

*Theorem:*

$$\boldsymbol{z} = \sum (uu^\dagger)\boldsymbol{z}_1 + \sum (\hat{u}\hat{u}^\dagger)\boldsymbol{z}_2$$

This decomposes *any* vector $\boldsymbol{z}$ into *six* eigenvectors, one for each eigenvalue of $\mathcal{A}$!

# OCTONIONIC EIGENVALUE PROBLEM

- Eigenvalues not necessarily real!
- New notion of orthogonality:

$$(vv^\dagger)w = 0$$

- 6 eigenvalues in $3 \times 3$ case!
- Decomposition:

$$\mathcal{A} = \sum \lambda v v^\dagger$$

# VECTORS II

$$\boldsymbol{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$\boldsymbol{X}^\dagger = \overline{\boldsymbol{X}}^T = \boldsymbol{X}$$

$$-\det(\boldsymbol{X}) = -t^2 + x^2 + y^2 + z^2$$

- {vectors in (3+1)-dimensional spacetime}
  $\longleftrightarrow$ {$2 \times 2$ complex Hermitian matrices}
- determinant $\longleftrightarrow$ (Lorentzian) inner product
- $\boldsymbol{X} = tI + x\sigma_x + y\sigma_y + z\sigma_z$ (Pauli matrices)

# LORENTZ TRANSFORMATIONS

Exploit (local) isomorphism:

$$SO(3,1) \approx SL(2,\mathbb{C})$$

$$\boldsymbol{x}' = \boldsymbol{\Lambda x} \qquad \longleftrightarrow \qquad \boldsymbol{X}' = \boldsymbol{M X M}^\dagger$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix}$$

$$\det(\boldsymbol{M}) = 1 \quad \Longrightarrow \quad \det \boldsymbol{X}' = \det \boldsymbol{X}$$

# WHICH DIMENSIONS?

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \longmapsto$

$$\boldsymbol{X} = \begin{pmatrix} p & \overline{a} \\ a & m \end{pmatrix} \qquad (p, m \in \mathbb{R}; a \in \mathbb{K})$$

$\dim \mathbb{K} + 2 = 3, 4, 6, 10$ spacetime dimensions

supersymmetry

$$SO(5,1) \approx SL(2,\mathbb{H})$$
$$SO(9,1) \approx SL(2,\mathbb{O})$$

# SPIN

**Finite rotation:**

$$R_z = \begin{pmatrix} e^{\ell\frac{\theta}{2}} & 0 \\ 0 & e^{-\ell\frac{\theta}{2}} \end{pmatrix}$$

**Infinitesimal rotation:**

$$L_z = \left.\frac{dR_z}{d\theta}\right|_{\theta=0} = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

**Right self-adjoint operator:**

$$\hat{L}_z\psi := (L_z\psi)\bar{\ell}$$

**Right eigenvalue problem:**

$$\hat{L}_z\psi = \psi\lambda$$

# ANGULAR MOMENTUM REVISITED

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \qquad L_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$L_z = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix} \qquad \hat{L}_\mu \psi := -(L_\mu \psi)\ell$$

$$[L_x, L_y] = -L_z \quad \ldots$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow$$

$$\hat{L}_z \psi = \psi\,\frac{1}{2} \qquad \hat{L}_x \psi = -\psi\,\frac{k}{2} \qquad \hat{L}_y \psi = -\psi\,\frac{k\ell}{2}$$

**Simultaneous eigenvector!**

(only 1 *real* eigenvalue)

# JORDAN ALGEBRAS

$$\mathcal{X} = \begin{pmatrix} p & a & \bar{c} \\ \bar{a} & m & b \\ c & \bar{b} & n \end{pmatrix}$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I} \end{aligned}$$

# JORDAN ALGEBRAS

$u, v, w \in \mathbb{R}^3 \Longrightarrow$

$$\begin{aligned} 2\,uu^\dagger \circ vv^\dagger &= (u \cdot v)\,(uv^\dagger + vu^\dagger) \\ \operatorname{tr}\left(uu^\dagger \circ vv^\dagger\right) &= (u \cdot v)^2 \\ 2\,uu^\dagger * vv^\dagger &= (u \times v)(u \times v)^\dagger \end{aligned}$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\operatorname{tr}(\mathcal{Y}) + \mathcal{Y}\operatorname{tr}(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\operatorname{tr}(\mathcal{X})\operatorname{tr}(\mathcal{Y}) - \operatorname{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I} \end{aligned}$$

# JORDAN ALGEBRAS

**Exceptional quantum mechanics:**

(Jordan, von Neumann, Wigner)

$$(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{X}^2 = \mathcal{X} \circ \left(\mathcal{Y} \circ \mathcal{X}^2\right)$$

$$\begin{aligned} \mathcal{X} \circ \mathcal{Y} &= \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}) \\ \mathcal{X} * \mathcal{Y} &= \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big(\mathcal{X}\,\mathrm{tr}\,(\mathcal{Y}) + \mathcal{Y}\,\mathrm{tr}\,(\mathcal{X})\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{tr}\,(\mathcal{X})\,\mathrm{tr}\,(\mathcal{Y}) - \mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I} \end{aligned}$$

# JORDAN EIGENVALUE PROBLEM

**Jordan product:**

$$\mathcal{A} \circ \mathcal{B} = \frac{1}{2}(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})$$

(**over** $\mathbb{R}$**:** $2\,uu^\dagger \circ vv^\dagger = (u \cdot v)\,(uv^\dagger + vu^\dagger)$)

**Eigenvalue problem:** (eigenmatrices!)

$$\mathcal{A} \circ \boldsymbol{x} = \lambda \boldsymbol{x}$$

- eigenvalues satisfy characteristic equation ($\mathcal{A}^3 \equiv \mathcal{A} \circ \mathcal{A} \circ \mathcal{A}!$)
- "only" 3 eigenvalues
- $\lambda \neq \mu \Longrightarrow \mathcal{V} \circ \mathcal{W} = 0$
- $\mathcal{A} = \sum \lambda \mathcal{V}$

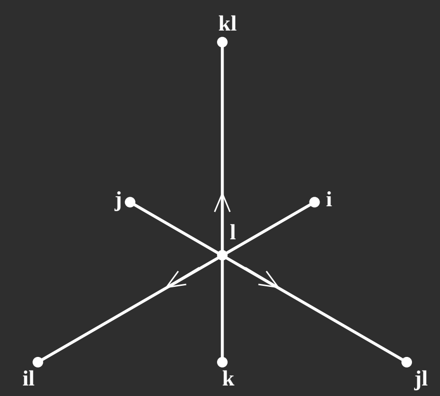
# DIMENSIONAL REDUCTION

$$SO(3,1) \subset SO(5,1) \subset SO(9,1)$$
$$SL(2,\mathbb{C}) \subset SL(2,\mathbb{H}) \subset SL(2,\mathbb{O})$$

**Mass is Quaternionic:**

$$P = \begin{pmatrix} p^t + p^z & p^x - \ell p^y - km \\ p^x + \ell p^y + km & p^t - p^z \end{pmatrix}$$

**How Many Quaternionic Spaces?**




**Answer:** 3!

**3 generations of leptons!**

**Neutrinos have just one helicity!**

# DIRAC EQUATION

$$\mathcal{P} = \begin{pmatrix} \boldsymbol{P} & \theta\xi \\ \bar{\xi}\theta^\dagger & |\xi|^2 \end{pmatrix}$$

$$\mathcal{P} * \mathcal{P} = 0 \Longrightarrow \widetilde{\boldsymbol{P}}\theta = 0$$

$$\mathcal{P} = \psi\psi^\dagger$$

**quaternionic!**

**Furthermore:** $$\mathcal{X}^\dagger = \mathcal{X} \Longrightarrow \mathcal{X} = \sum_{n=1}^{3} \psi_n \psi_n^\dagger$$

**leptons, mesons, baryons?**

**Life is complex.**

It has real and imaginary parts.

**Life is octonionic...**

**THE END**