The OCTONIONIC EIGENVALUE Problem *REVISITED*

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I: Eigenvectors
II: Octonionic Eigenvectors
III: Octonionic Projections
IV: "Almost Quaternions"

DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$
$$z = x + yi$$

Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$
$$q = (a + bi) + (c + di)j$$

Octonions:

$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

COMPLEX EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = \lambda v$

Hermitian: $A^\dagger = \overline{A}^T = A$

Reality: $\overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v$

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

since: $\lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 = v_1^\dagger A v_2 = v_1^\dagger \lambda_2 v_2$

- $\exists\, n$ eigenvalues, all real.
- $\exists$ basis of orthonormal eigenvectors.

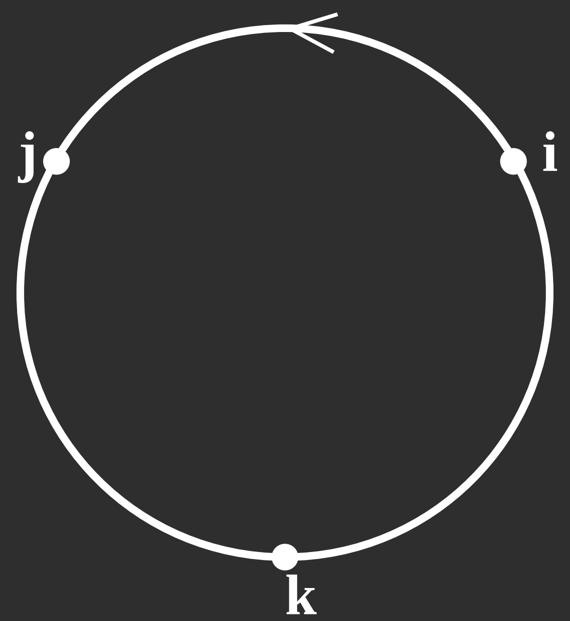
Decomposition:

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$$

QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

not commutative



QUATERNIONIC EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = \lambda v$

Hermitian: $A^\dagger = A$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 - j \\ k - i \end{pmatrix} = (1 - j) \begin{pmatrix} 1 \\ k \end{pmatrix}$$

Eigenvalues need not be real!

Furthermore, $A(qv) \neq q(Av)$, so:

Multiples of eigenvectors not eigenvectors!

RIGHT EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = v\lambda$

Hermitian: $A^\dagger = A$

Reality:

$$\begin{aligned}\overline{\lambda}(v^\dagger v) &= (\overline{\lambda}v^\dagger)v = (Av)^\dagger v = (v^\dagger A)v \\ &= v^\dagger(Av) = v^\dagger(v\lambda) = (v^\dagger v)\lambda\end{aligned}$$

Multiples:

$$A(vq) = (Av)q = (v\lambda)q = vq\lambda$$

- $\exists\, n$ eigenvalues, all real.
- $\exists$ basis of orthonormal eigenvectors.

Decomposition:

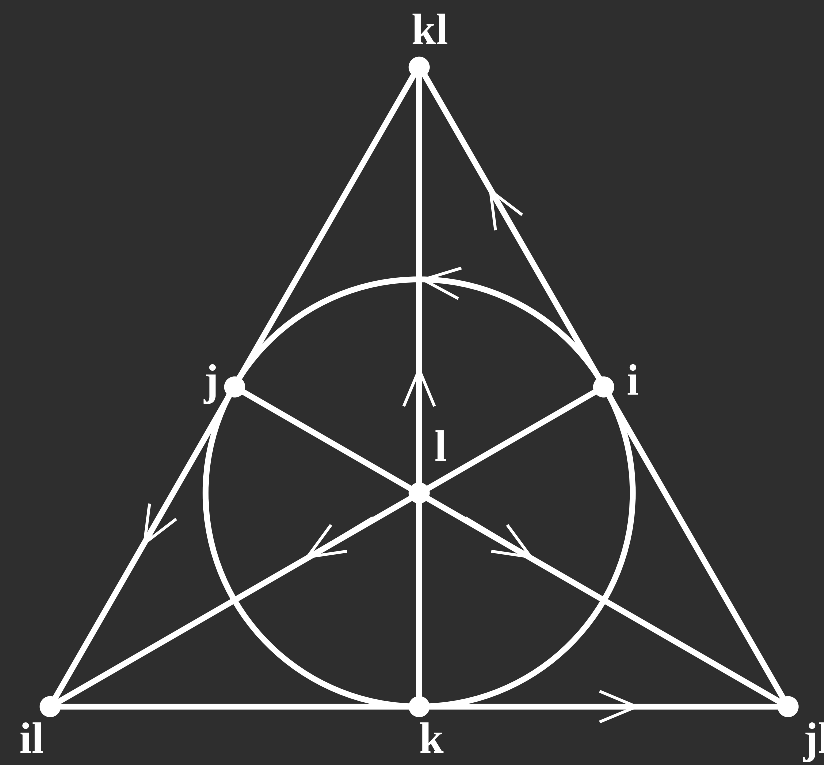
$$\boxed{A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger}$$

OCTONIONS

each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative



WHAT STILL WORKS?

- $q \in \operatorname{Im} \mathbb{O} \Longrightarrow \bar{q} = -q$
- $q\bar{q} = |q|^2$
- $q^{-1} = \dfrac{\bar{q}}{|q|^2} \qquad (q \neq 0)$
- $|pq| = |p||q|$
- $[p, p, q] = (pp)q - p(pq) = 0$

alternativity

OCTONIONIC EIGENVALUE PROBLEM

Eigenvalue Equation: $Av = v\lambda$

Hermitian: $A^\dagger = A$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} j \\ \ell \end{pmatrix} = \begin{pmatrix} j + i\ell \\ \ell - k \end{pmatrix} = \begin{pmatrix} j \\ \ell \end{pmatrix} (1 - k\ell)$$

Eigenvalues need not be real!

$$\begin{aligned} \overline{\lambda}(v^\dagger v) &\neq (\overline{\lambda} v^\dagger) v = (Av)^\dagger v = (v^\dagger A) v \\ &\neq v^\dagger (Av) = v^\dagger (v\lambda) \neq (v^\dagger v)\lambda \end{aligned}$$

3×3 REAL EIGENVALUE PROBLEM

Characteristic Equation:

$$\mathcal{A}^3 - (\operatorname{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det\mathcal{A})\,I = 0$$

$$\mathcal{A}^3 = \frac{1}{2}(\mathcal{A}\mathcal{A}^2 + \mathcal{A}^2\mathcal{A})$$

But:

$$\lambda^3 - (\operatorname{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - \det\mathcal{A} = r$$

$$(r - r_+)(r - r_-) = 0$$

- 2 sets of 3 real eigenvalues!
- 4-parameter families of eigenvectors
- $\det\mathcal{A} = 0 \quad \not\Longrightarrow \quad \lambda = 0$!

DECOMPOSITIONS I

Family: $K[v] = rv =$

$$\mathcal{A}\Big(\mathcal{A}\big(\mathcal{A}(v)\big)\Big) - (\mathrm{tr}\mathcal{A})\,\mathcal{A}\big(\mathcal{A}(v)\big) + \sigma(\mathcal{A})\,\mathcal{A}(v) - \det(\mathcal{A})\,v$$

Theorem: (v, w in same family; $\lambda \neq \mu$)

$$(vv^\dagger)w = 0$$

Proof: 8 hour brute force Mathematica computation!

(Analytic proof by Okubo!)

Corollary:

$$\mathcal{A} = \sum \lambda v v^\dagger$$

OCTONIONIC EIGENVALUE PROBLEM

- Eigenvalues not necessarily real!
- New notion of orthogonality:

$$(vv^\dagger)w = 0$$

- 6 eigenvalues in 3×3 case!
- Decomposition:

$$\mathcal{A} = \sum \lambda v v^\dagger$$

PROJECTIONS

Idea: $(uu^\dagger)\,\boldsymbol{z} \longleftrightarrow u\,(u \cdot \boldsymbol{z})$

$(uu^\dagger + vv^\dagger + ww^\dagger = I)$

Theorem: ($\boldsymbol{z}$ *any* vector in same family)

$$(uu^\dagger)\Big((uu^\dagger)\boldsymbol{z}\Big) = (uu^\dagger)\boldsymbol{z}$$

Corollary:

$$(vv^\dagger)\Big((uu^\dagger)\boldsymbol{z}\Big) = 0$$

Corollary:

$$\mathcal{A}\Big((uu^\dagger)\boldsymbol{z}\Big) = \lambda\Big((uu^\dagger)\boldsymbol{z}\Big)$$

DECOMPOSITIONS II

Into Families:

$$\boldsymbol{z} = \frac{K[\boldsymbol{z}] - r_2\,\boldsymbol{z}}{r_1 - r_2} - \frac{K[\boldsymbol{z}] - r_1\,\boldsymbol{z}}{r_1 - r_2} = \boldsymbol{z}_1 + \boldsymbol{z}_2$$

Within Families: $(uu^\dagger + vv^\dagger + ww^\dagger = I)$

$$\boldsymbol{z}_1 = (uu^\dagger)\boldsymbol{z}_1 + (vv^\dagger)\boldsymbol{z}_1 + (ww^\dagger)\boldsymbol{z}_1$$

Theorem:

$$\boldsymbol{z} = \sum (uu^\dagger)\boldsymbol{z}_1 + \sum (\hat{u}\hat{u}^\dagger)\boldsymbol{z}_2$$

This decomposes *any* vector $\boldsymbol{z}$ into *six* eigenvectors, one for each eigenvalue of $\mathcal{A}$!

EXAMPLE: $\mathcal{A}$ quaternionic

Orthonormal basis of eigenvectors:

$$\mathcal{A}v = v\lambda \qquad \mathcal{A} = \sum \lambda v v^\dagger \qquad \mathcal{I} = \sum v v^\dagger$$

Octonionic eigenvectors: $(\overline{\mathcal{A}}w = w\mu)$

$$\mathcal{A}(\ell w) = \ell(\overline{\mathcal{A}}w) \Longrightarrow \mathcal{A}(\ell w) = (\ell w)\mu$$

$$\Longrightarrow \mathcal{A} = \sum \mu(\ell w)(\ell w)^\dagger \qquad \mathcal{I} = \sum (\ell w)(\ell w)^\dagger$$

Projection:

$$\begin{aligned} u &= u_1 + \ell u_2 \\ &= \sum (v v^\dagger) u_1 + \sum \left((\ell w)(\ell w)^\dagger\right)(\ell u_2) \\ &= \sum v v^\dagger u_1 + \sum \ell(w w^\dagger u_2) \end{aligned}$$

“ALMOST QUATERNIONS”

Families of vectors: $(K[v] = rv,\ r = r_\pm)$ K diagonal!

Families of octonions: $(K[q] = rq)$

Decompositions of $\mathbb{O}$: $(\mathbb{T} = \langle 1, a, b, c \rangle)$

$$\mathbb{O} = \mathbb{T} \oplus \mathbb{T}[a, b, c]$$

$$p, q \in \mathbb{T}s \Longrightarrow p\bar{q} \in \mathbb{T}$$

components of $vv^\dagger$ are in $\mathbb{T}$!

Depends only on $\mathbb{T}$, not on choice of a, b, c!

THE END