The Purpose of Education

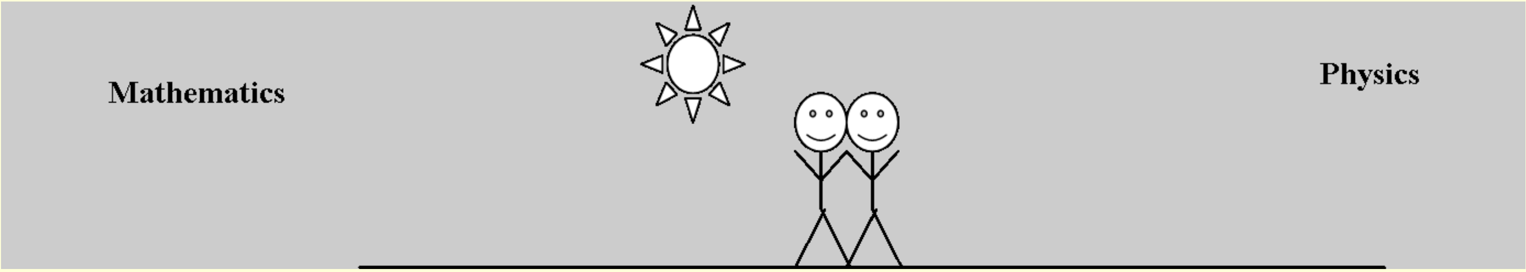
"For an educational process to be truly successful, it must encourage students to reflect on the conceptual foundation of their own education."
—FUNDAEC

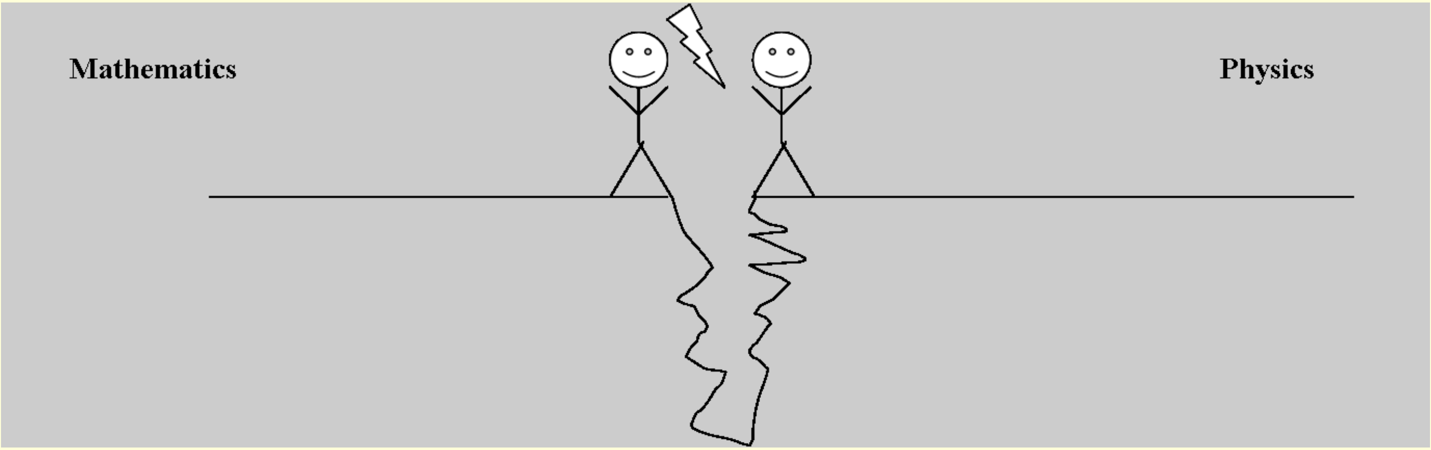
Fundación para la Aplicación y Enseñanza de las Ciencias

Bridging the Gap:
Vector Calculus in Mathematics and Physics

Tevian Dray & Corinne Manogue

Mathematics vs. Physics





Goals for this Talk

Show how geometry helps mathematics problem solving.

Explain WHY it is so hard to apply your mathematics knowledge in physics.

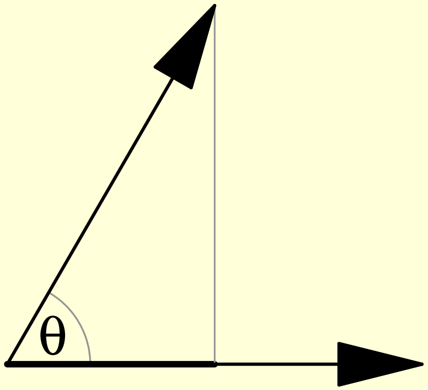
Concept Image

Concept Image: the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes.

Tall and Vinner, *Educational Studies in Mathematics,* (1981).

Tell me something you know about the dot product.

Explain your answer to your neighbor.

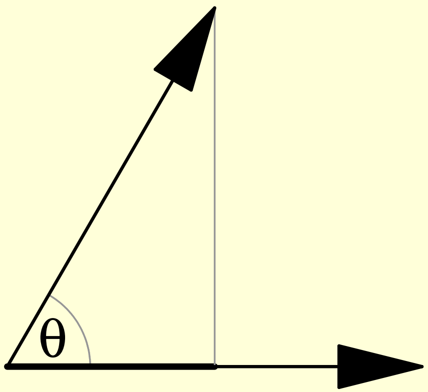


Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$

Dot Product



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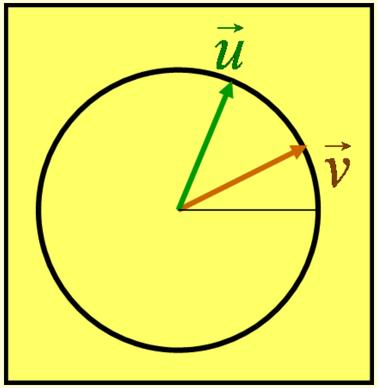


Addition Formulas:

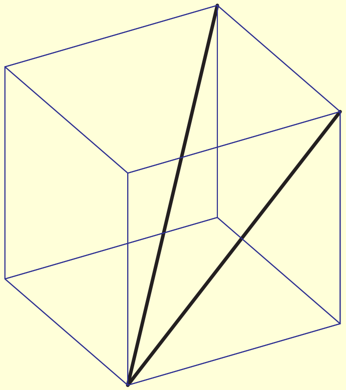
$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\mathbf{x}} + \sin\beta\,\hat{\mathbf{y}}$$

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \cos(\alpha - \beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

Cube

Find the angle between the diagonal of a cube and the diagonal of one of its faces.

(Work with your neighbors.)



Algebra:

$$\begin{aligned}\vec{\mathbf{u}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\ \vec{\mathbf{v}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}}\end{aligned}$$

$$\Longrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2$$

Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta$$

$$\therefore \cos\theta = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{1}{\sqrt{6}}$$

Need both!

Concept Image of Derivative

Tell me something you know about the derivative.

Explain your answer to your neighbor.

Concept Image of 1-D Derivative

Ratio

Slope

Limit

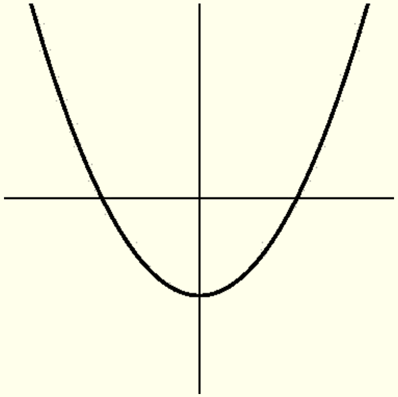
Function

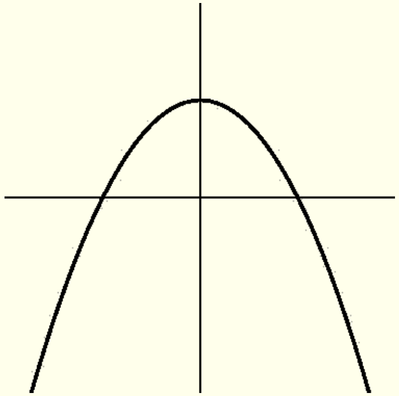
Rate of Change

Velocity

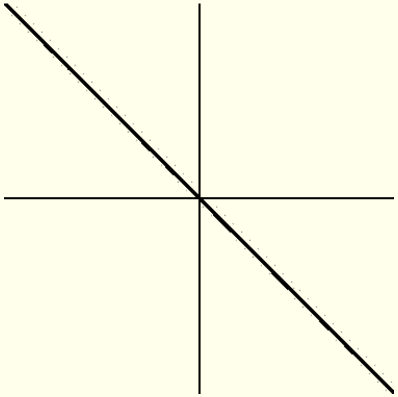
Difference Quotient

- Which of the graphs below could represent the derivative of the function which is graphed at the right?





A



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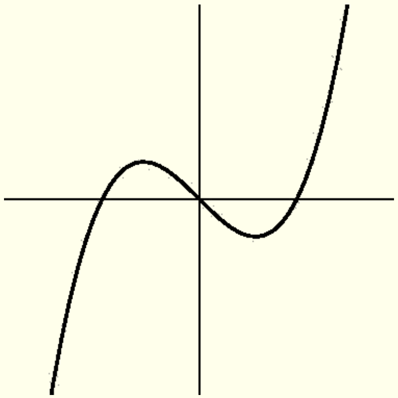
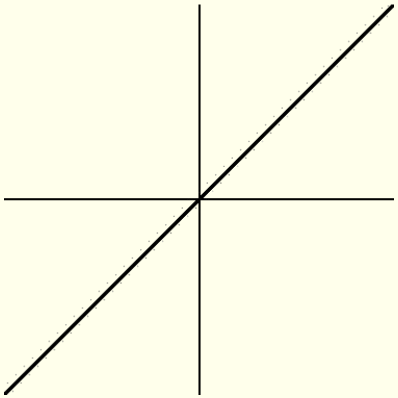


C



D

Gradient

Tell me something you know about the gradient.

Explain your answer to your neighbor.

- The gradient points in the steepest direction.
- The magnitude of the gradient tells you how steep.
- $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \dots$

Gradient

Chain Rule:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}}\right) \cdot \left(\frac{dx}{dt}\,\hat{\mathbf{x}} + \frac{dy}{dt}\,\hat{\mathbf{y}}\right) \\ &= \vec{\nabla} f \cdot \vec{\mathbf{v}}\end{aligned}$$

$$f = \text{const} \Longrightarrow \frac{df}{dt} = 0 \;\Longrightarrow\; \vec{\nabla} f \perp \vec{\mathbf{v}}$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{\vec{\mathbf{v}}}{\vec{\mathbf{v}}}$$

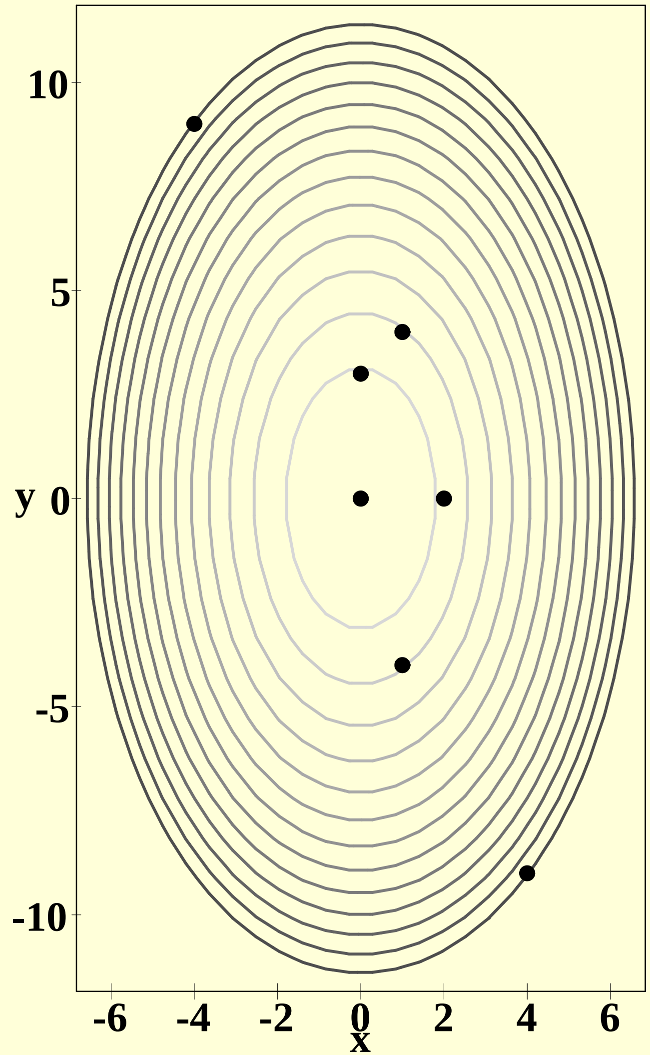
The gradient points in the steepest direction

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

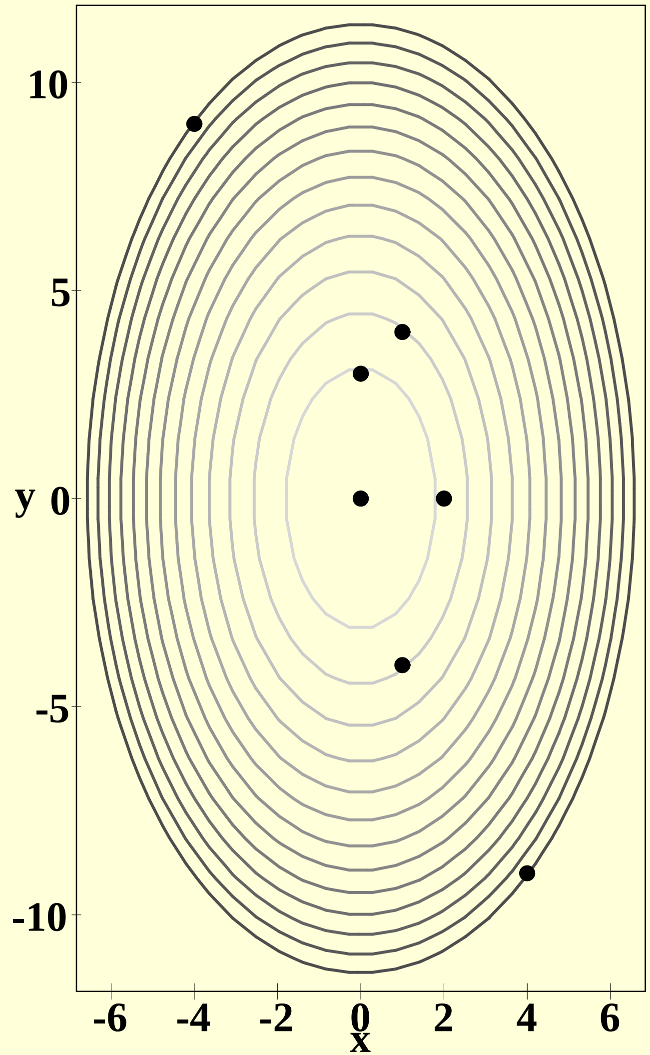
where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



The Hill

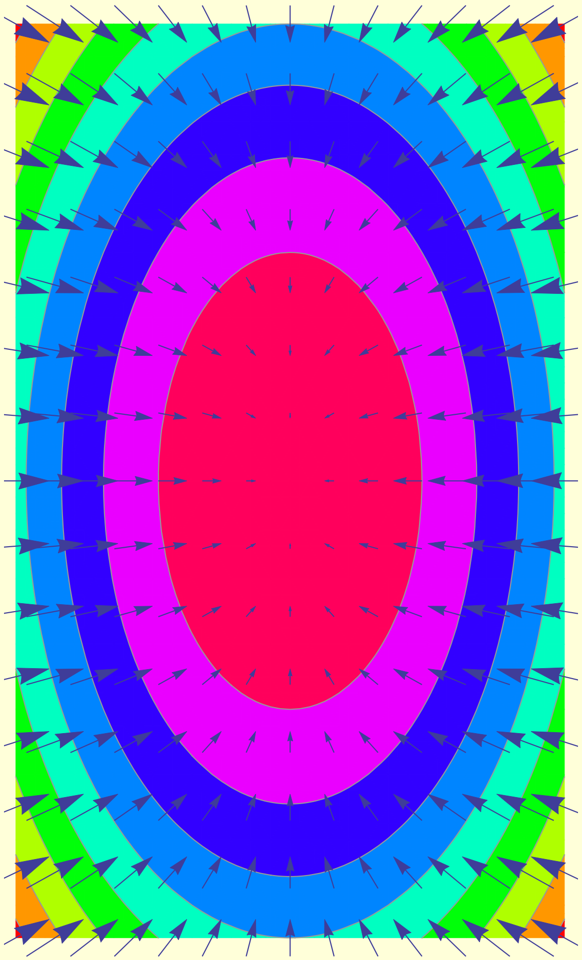
Stand up and close your eyes.

Hold out your right arm in the direction of the gradient where you are standing.



Gradient: Which Direction?

Visualization



Introductory Mechanics:

Vector-valued functions of time.

$$\vec{x}(t)$$

$$\vec{v}(t) = \frac{d}{dt}\vec{x}(t)$$

$$\vec{a}(t) = \frac{d^2}{dt^2}\vec{x}(t)$$

Concept Image of Divergence

Tell me something you know about divergence.

Explain your answer to your neighbor.

Divergence in Mathematics

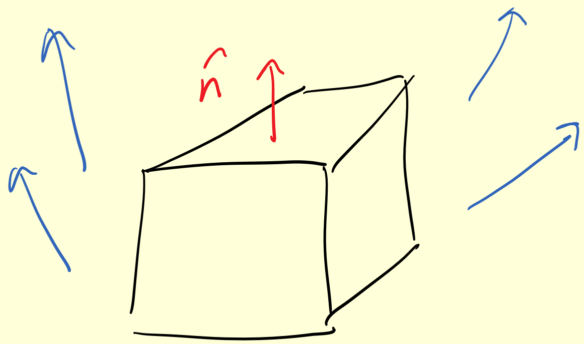
Definition in rectangular coordinates
(Note algebra, not geometry!)

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$div\ F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Divergence: Geometric Definition

$$\left.\frac{\left(\left(\vec{\mathbf{F}}\cdot\hat{\mathbf{n}}\right)\,dx\,dy\right)\,dz}{dz}\right|_{\text{top}} + \left.\frac{\left(\left(\vec{\mathbf{F}}\cdot\hat{\mathbf{n}}\right)\,dx\,dy\right)\,dz}{dz}\right|_{\text{bot}} = \frac{\partial F_z}{\partial z}d\tau$$

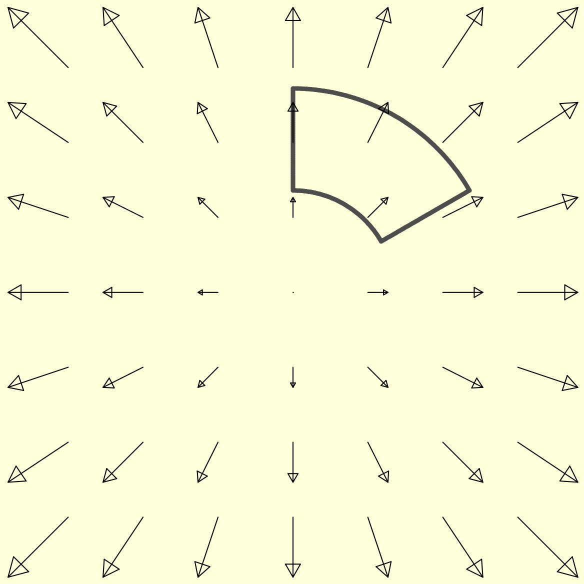


Flux per unit volume
Schey, *div, grad, curl and all that*, Dover

Coordinate independence of definition

$$\vec{\mathbf{F}} = r\,\hat{\mathbf{r}}$$

$$\vec{\nabla}\cdot\vec{\mathbf{F}} = \frac{1}{r}\frac{\partial}{\partial r}(r\,F_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$



Divergence and Gauss

Add a physics law.

$$\oint_{\substack{closed\\surface}} \vec{E}\cdot\hat{n}\,dA = \frac{Q_{enclosed}}{\varepsilon_0}$$

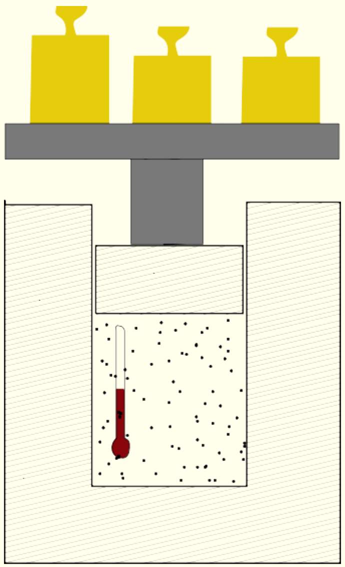
$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Thermodynamics: State Variables

Name the experiment:

$$\left(\frac{\partial p}{\partial T}\right)_S$$

Name the Experiment



$$\left(\frac{\partial p}{\partial T}\right)_S \quad vs. \quad \left(\frac{\partial p}{\partial T}\right)_V$$

Learning Strategies

Use geometry

Build rich concept images

Also:

Use available resources

Have confidence—especially important for women and underrepresented groups.

Use Available Resources

Go to class! Ask questions.

Tutoring

Form study groups

Find mentors

Mentoring

Find a mentor:

- Find many mentors for the different aspects of yourself.
- Just because you ask for advice doesn't mean you need to take it.

Be a mentor:

- Listen carefully and echo back what the person is saying. Give open ended advice that leaves the person options.
- Assume the person doesn't know how to do what you suggest.
- "He imagined things for me that I could not imagine for myself."
- Consider the department climate for women.

Have Confidence (even if you pretend)

Understand stereotype threat.

Use the Super Woman pose.

Change fixed mindset to growth mindset:

- Make new neurons
- Active learning

Beware the imposter/Curie syndrome.

Learn to brag in a culturally appropriate way.

Make use of resources available to you.

Handling negativity

Don't reject yourself:

- No one who has accepted you wants you to fail. (People get paid to reject you, make them earn their money.)

Don't believe the statistics:

- Women are less smart at math-> geometry -> 3d -> women can learn.

Don't automatically accept statements from others:

- "You don't need that job."
- Proving over and over again.
- "Are you the secretary or the genius?"
- "You can have THIS woman faculty member."

ISRO's Female Scientists

After the success of the Mars Orbiter Mission on 24 September 2014. Women are 20% of the space agency's total workforce.

Geometry, geometry, geometry...

Tevian Dray & Corinne A. Manogue

http://math.oregonstate.edu/bridge
http://physics.oregonstate.edu/portfolioswiki