Using Active Engagement to Teach Mathematics

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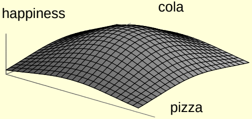
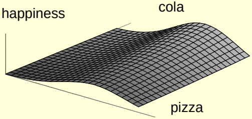
Pizza

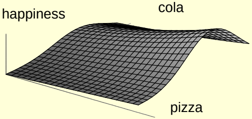
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if *there is such a thing as too many pizzas but no such thing as too much cola*?



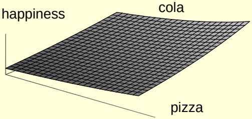
A



B



C



D

DO NOT VOTE UNTIL TOLD TO DO SO!

(Ne votez pas avant qu'on vous le dise!)

My Background

- Undergraduate degree in mathematics. (Only.)
- Doctorate in mathematics. (Relativity!)
- Postdocs in both mathematics and physics.
- My wife is a physicist. (Double degree in physics and math.)
- We work together. (30 articles & 2 books; math, physics, ed.)
- Each of us is a Fellow of the American Physical Society.
- We have each won a national teaching award.
- Our daughter is a math educator. (Also double degree.)

My department thinks I'm a physicist.
(The physics department knows better.)

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Outline

I: Practice

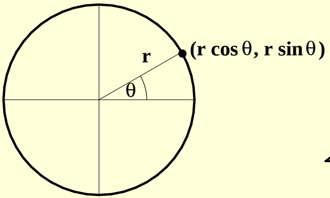
II: Theory

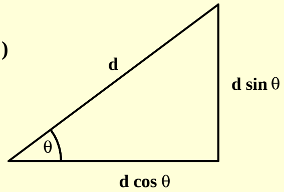
Tell me something you know about trigonometry.

(Write your answer on your small whiteboard.)

Dites-moi quelques choses à propos de la trigonométrie.

(Écrivez votre réponse sur votre tableau blanc.)





$$\cos^2\theta + \sin^2\theta = 1$$

▸ Circle Trig

Small WhiteBoard Questions (SWBQs)

Things to consider:

- Open-ended.
- Recollection is more challenging than recognition.

Classroom implementation:

- Everyone must write something – but "??" OK.
- Gather responses and discuss. (Anonymize!)
- Can be spontaneous.

Things to consider:

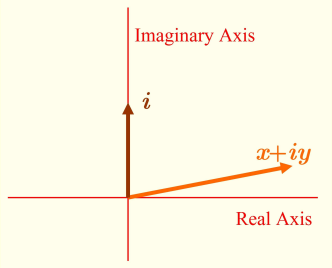
- Whenever possible, base your instruction on what is known about incoming student resources.
- Example: Dr. Emily Smith (OSU 2016) showed that many upper-division physics students know triangle trigonometry, but not unit-circle trigonometry. This causes problems with complex numbers.

Classroom implementation:

- "Review" circle trigonometry before using it
- One possibility: Use simulation.

Complex Plane

$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$



$$\boxed{i^2 = -1}$$

$$(x, y) \longmapsto x + iy$$

$$x + iy = r\,\cos\theta + i\,r\,\sin\theta = r\,e^{i\theta}$$

Special case: $e^{\pm i\pi/2} = \pm i$

$$\boxed{e^{i\pi} + 1 = 0}$$

Representing Complex Numbers

- Please stand up. (Levez-vous s'il vous plaît.)
- Use your left hand. (Utilisez votre main gauche.)
- Real axis points forward. (L'axe réel pointe en avant.)
- Imaginary axis points upward. (L'axe imaginaire est vers le haut.)

Show me:

(Montrez moi:)

- 1
- $2\,i$
- $1 + i$
- $e^{-i\pi/3}$

Kinesthetic Activity

Things to consider:

- Everyone is awake!
- Teacher can see what everyone is thinking.
- Highlights geometric reasoning.
- Students get geometric cues from others.
- Students must make a decision.
- Student can be asked to translate representations.

Classroom implementation:

- *Please stand up.*
- *Show me...*
- *Thank you, you can sit down.*

Multiplication by i

$$(1+i)i = i - 1$$

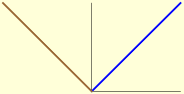
If $1+i$ is multiplied by i, the corresponding vector is:

A: Reflected about the x-axis

B: Reflected about the y-axis

C: Rotated by $\frac{\pi}{2}$ (90°) counterclockwise

D: Rotated by $\frac{\pi}{2}$ (90°) clockwise



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Concept Tests/Peer Instruction/Clickers

Things to consider:

- Asks students to make a commitment.
- Asks students to defend an answer.
- Good questions: conceptual, focus on common mistakes.

Classroom implementation:

- Many "response" systems:
  clickers, ABCD cards, whiteboards, fingers.
- Two stages.
- Simultaneous and anonymous.
- *Convince your neighbor.*

Multiplication by i

$$(1 + 2i)i$$

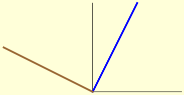
If $1 + 2i$ is multiplied by i, the corresponding vector is:

A: Reflected about the x-axis

B: Reflected about the y-axis

C: Rotated by $\frac{\pi}{2}$ (90°) counterclockwise

D: Rotated by $\frac{\pi}{2}$ (90°) clockwise



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Multiplication by i

$$\boxed{(re^{i\theta})i}$$

If $re^{i\theta}$ is multiplied by i, the corresponding vector is

A: Reflected about the x-axis

B: Reflected about the y-axis

C: Rotated by $\frac{\pi}{2}$ (90°) counterclockwise

D: Rotated by $\frac{\pi}{2}$ (90°) clockwise

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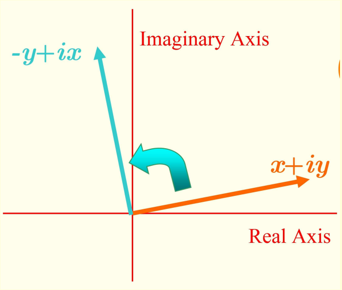
Multiplication by i



Multiplication by i: $(x + i\,y)i = i\,x + i^2 y = -y + ix$

Rotates counterclockwise by $\pi/2$

Multiplication by $s\,e^{i\alpha}$: $(r\,e^{i\theta})(s\,e^{i\alpha}) = rs\,e^{i(\theta+\alpha)}$

Rotates counterclockwise by α
and stretches by s

Sequences of Questions

Things to consider:

- Frame the sequence with increasing sophistication.
- Choose clicker questions vs. SWBQs by need for open-endedness.
- Choose clicker questions vs. SWBQs by type of response desired.

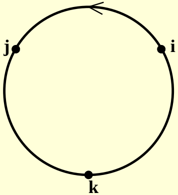
Classroom implementation:

- Use wrap-up as an opportunity for reflection.

(SWBQ = Small WhiteBoard Question)

Quaternions

$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$



$$q = (x + y\,i) + (z + w\,i)\,j = x + y\,i + z\,j + w\,k$$

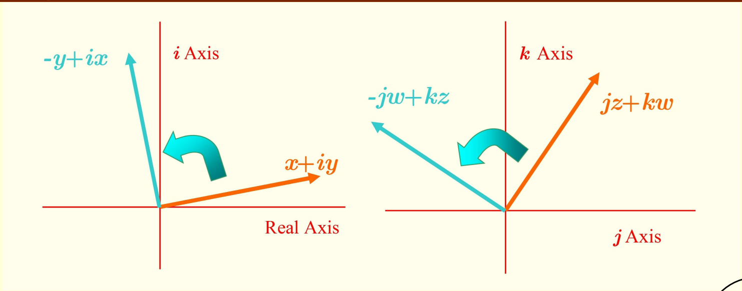
$$\boxed{ij = k = -ji;\, i^2 = j^2 = k^2 = -1}$$

$\mathbb{H}$ is for Hamilton! ($\mathbb{Q}$ denotes rationals)

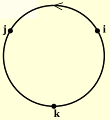
Calculate with your group: *iq* and *qi*

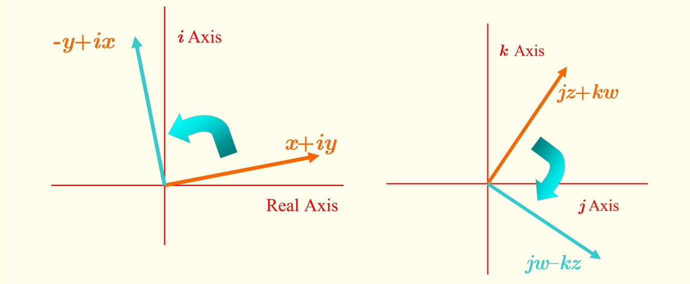
(Calculer avec votre groupe: *iq* et *qi*)

iq vs. qi



$$
\begin{aligned}
q &= x + iy + jz + kw \\
iq &= ix - y + kz - jw \\
qi &= ix - y - kz + jw
\end{aligned}
$$





Small Group Activity

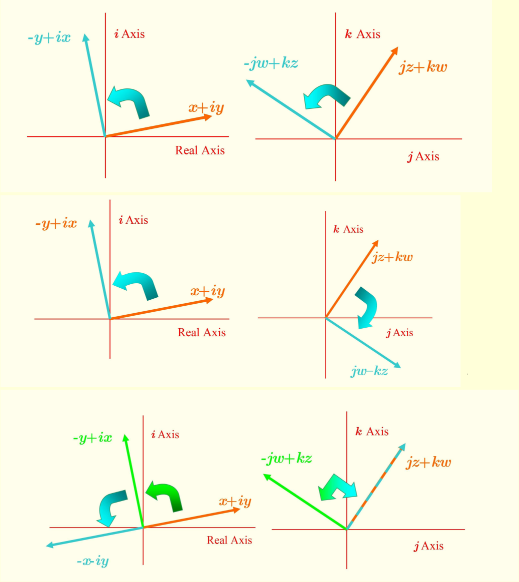
Things to consider:

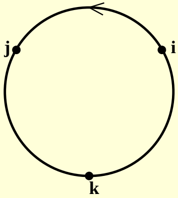
- Can emphasize more complex problems/reasoning.
- Students practice problem solving themselves.
- Equity: moves office hours into the classroom.

Classroom implementation:

- *You have 10 minutes; GO!*
- Who needs help?
- Do you need more time?
- Pause.

Conjugation





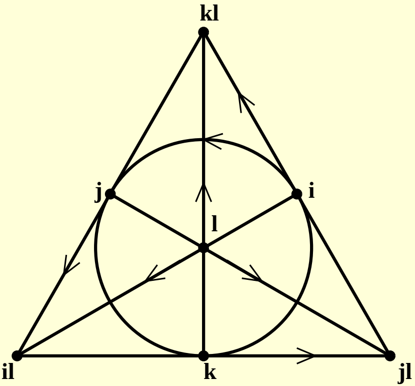
$$q = x + iy + jz + kw$$

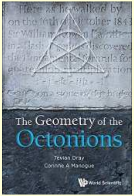
$$iq = ix - y + kz - jw$$
$$qi = ix - y - kz + jw$$

$$iqi = -x - iy + jz + kw$$
$$-iqi = x + iy - jz - kw$$
(rotation in jk-plane)

Things to consider:

- Lecture is fast; use it when it works.
- What is the focus of attention? (You, the slides, their notes...)
- How busy are the slides?
- Do the figures have distracting elements?





2015

Octonions! $(\mathbb{O} = \mathbb{H} + \mathbb{H}\ell)$

Use to model particle physics

http://octonions.geometryof.org/GO

Plum Muffins

Story telling is memorable.

SUMMARY #1: Lecture (vs. Activities)

The Instructor:

- Paints big picture
- Inspires.
- Covers lots fast.
- Models speaking.
- Models problem-solving.
- Controls questions.
- Makes connections.
- Demonstrates new complicated reasoning.

The Students:

- Focus on subtleties.
- Experience delight.
- Slow, but in depth.
- Practice speaking.
- Practice problem-solving.
- Control questions.
- Make connections.
- Discover questions about what is complicated.

Mathematics vs. Physics

Is there a difference between $\frac{x^2 - 4}{x - 2}$ and $x + 2$?

Mathematics and Physics are two disciplines separated by a common language!

Physicists are bilingual (but don't know it)

What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?

Share your answer with your neighbor(s).
(Discutez avec votre voisin.)

A: $T(r, \theta) = kr^2$
B: $T(r, \theta) = k(r^2 + \theta^2)$

Are mathematicians bilingual?

Theoretical background

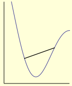
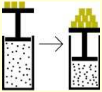
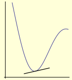
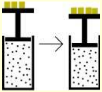
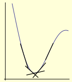
- **Vinner (1983):** A *concept image* is the set of properties associated with a concept together with the mental pictures of the concept.
- **Sfard (1991):** The *process-object* framework describes mathematics as proceeding through processes acting on objects, with those processes then becoming reified into objects.
- **Zandieh (2000):** Student understanding of the concept of derivative can be described by associating *process-object layers* with *representations* or *contexts*.

Zandieh (2000)

Process-object layer	Graphical	Verbal	Physical	Symbolic	Other
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Michelle Zandieh, *A theoretical framework for analyzing student understanding of the concept of derivative*, CBMS Issues in Mathematics Education **8**, 103–122, 2000.

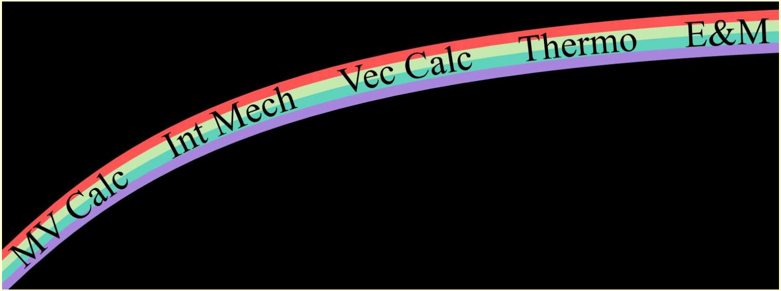
Extended Theoretical Framework for Concept of Derivative

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"avg. rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ numerically	
Limit		"inst. rate of change"	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"...at any point/time"	$f'(x) =$ ...	... depends on x	tedious repetition

No entry for symbolic differentiation!!

Roundy, Dray, Manogue, Wagner, & Weber, CRUME 18 Proceedings, MAA, 2015. `http://sigmaa.maa.org/rume/Site/Proceedings.html`

Learning Progression



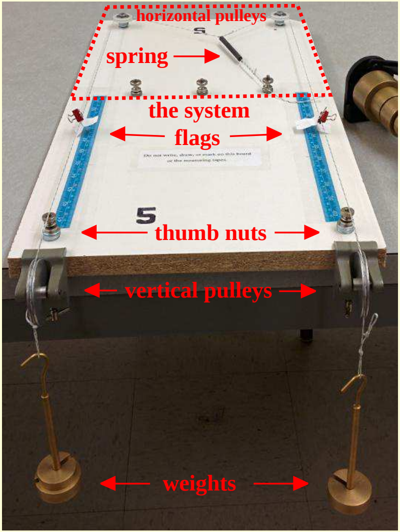
- Successively more sophisticated ways of thinking about a topic.
- Sequences supported by research on learner's ideas and skills.
- *Lower anchor* grounded in students' prior ideas and skills.
- *Upper anchor* grounded in knowledge and practices of experts.

Duschle et al., NRC, 2007; Plummer, 2012; Sikorski et al., 2009, 2010
Manogue, Dray, Emigh, Gire, & Roundy, PERC 2017

Partial Derivative Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.
- Mathematicians:
  "That's not a derivative!"

Roundy et al., *Experts' Understanding of Partial Derivatives Using the Partial Derivative Machine*, PERC 2014



(Each surface is dry-erasable, as are the matching contour maps.)

Raising Calculus to the Surface (Aaron Wangberg)

Raising Physics to the Surface (+ Liz Gire, Robyn Wangberg)

http://raisingcalculus.winona.edu

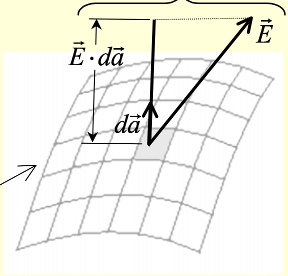
Multiple Representations

Flux is the total amount of electric field through a given area.

Perpendicular Component

Normal Vector

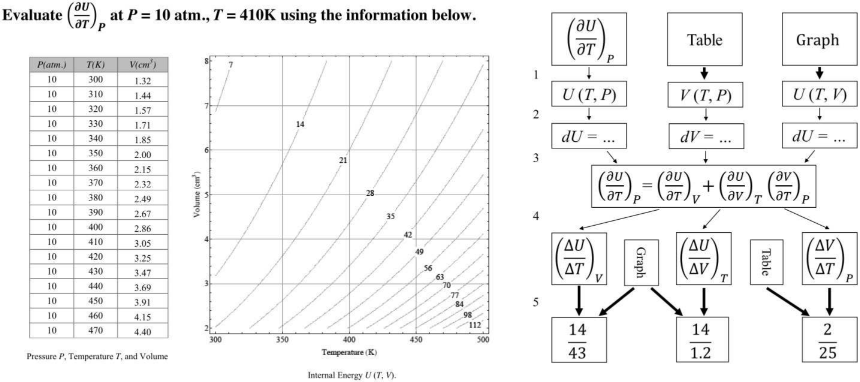
$$\Phi = \int \vec{E} \cdot d\vec{a}$$



$\sum$ over all rectangles

Kerry Browne (Ph.D. 2002)

Representational Transformation



P(atm.)	T(K)	V(cm³)
10	300	1.32
10	310	1.44
10	320	1.57
10	330	1.71
10	340	1.85
10	350	2.00
10	360	2.15
10	370	2.32
10	380	2.49
10	390	2.67
10	400	2.86
10	410	3.05
10	420	3.25
10	430	3.47
10	440	3.69
10	450	3.91
10	460	4.15
10	470	4.40

Rabindra R. Bajracharya, Paul J. Emigh, and Corinne A. Manogue, *Students' stategies for solving a multi-representational partial derivative problem in thermodyanmics*, in preparation.

SUMMARY II: Teaching Geometric Reasoning

Vector Calculus Bridge Project:

http://math.oregonstate.edu/bridge

- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (http://math.oregonstate.edu/BridgeBook)

Paradigms in Physics Project:

http://physics.oregonstate.edu/portfolioswiki

- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*