THE GEOMETRY OF SPECIAL RELATIVITY

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8 July 2003

Lorentz transformations are just hyperbolic rotations.

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Preface

The unification of space and time introduced by Einstein's special theory of relativity is one of the cornerstones of the modern scientific description of the universe. Yet the unification is counterintuitive, since we perceive time very differently from space. And, even in relativity, time is not just another dimension, it is one with different properties. Some authors have tried to "unify" the treatment of time and space, typically by replacing t by it, thus hiding some annoying minus signs. But these signs carry important information: Our universe, as described by relativity, is *not* Euclidean.

This short book treats the geometry of hyperbolas as the key to understanding special relativity. This approach can be summarized succinctly as the replacement of the ubiquitous γ symbol of most standard treatments with the appropriate hyperbolic trigonometric functions. In most cases, this not only simplifies the appearance of the formulas, but emphasizes their geometric content in such a way as to make them almost obvious. Furthermore, many important relations, including but not limited to the famous relativistic addition formula for velocities, follow directly from the appropriate trigonometric addition formulas.

I am unaware of any other introductory book on special relativity which adopts this approach as fundamental. Many books point out the relationship between Lorentz transformations and hyperbolic rotations, but few actually make use of it. A pleasant exception was the original edition of Taylor and Wheeler's marvelous book [1], but much of this material was removed from the second edition [2].

At the same time, this book is not intended as a replacement for that or any of the other excellent textbooks on special relativity. Rather, it is intended as an introduction to a particularly beautiful way of looking at special relativity, in hopes of encouraging students to see beyond the formulas to the deeper structure. Enough applications are included to get the basic idea, but these would probably need to be supplemented for a full course.

While much of the material presented can be understood by those familiar with the ordinary trigonometric functions, occasional use is made of elementary differential calculus. In addition, the chapter on electricity and magnetism assumes the reader has seen Maxwell's equations, and has at least a passing acquaintance with vector calculus. A prior course in calculus-based physics, up to and including electricity and magnetism, should provide the necessary background.

After a general introduction in Chapter 1, the basic physics of special relativity is described in Chapter 2. This is a quick, intuitive introduction to special relativity, which sets the stage for the geometric treatment which follows. Chapter 3 summarizes some standard (and some not so standard) properties of ordinary 2-dimensional Euclidean space, expressed in terms of the usual circular trigonometric functions; this geometry will be referred to as *circle geometry*. This material has deliberately been arranged so that it closely parallels the treatment of 2-dimensional Minkowski space in Chapter 4 in terms of hyperbolic trigonometric functions, which we call *hyperbola geometry*. [1] Special relativity is covered again from the geometric point of view in Chapter 5, which is followed by a discussion of some of the standard "paradoxes" in Chapter 8, applications to relativistic mechanics in Chapter 9, and the relativistic unification of electricity and magnetism in Chapter 11. Finally, Chapter 13 contains a brief discussion of the further steps leading to Einstein's general theory of relativity.

[1] Not to be confused with *hyperbolic geometry*, the curved geometry of the 2-dimensional unit hyperboloid. See Chapter 13.

Acknowledgments

This book grew out of class notes for a course on *Reference Frames*, which in turn forms part of a major upper-division curriculum reform effort, entitled *Paradigms in Physics*, which was begun in the Department of Physics at Oregon State University in 1997. I am grateful to all of the faculty involved in this effort, but especially to the leader of the project, Corinne Manogue, for support and encouragement at every stage. The *Paradigms in Physics* project was supported in part by NSF grant DUE–965320, supplemented with funds from Oregon State University; my own participation was made possible thanks to the (sometimes reluctant!) support of my department chair, John Lee. I was fortunate in having excellent teaching assistants, Jason Janesky and Emily Townsend, the first times I taught the course. A course based on an early draft of this book was taught at Mount Holyoke College in 2002, giving me an opportunity to make further revisions; my stay at Mount Holyoke was partially supported by their Hutchcroft Fund. I am grateful to Greg Quenell for having carefully read the manuscript at that time, and for suggesting improvements. Last but not least, I thank the many students who struggled to learn physics from a mathematician, enriching all of us.

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