The Octonionic Eigenvalue Problem

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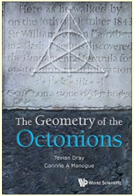
Book



The Geometry of the Octonions
Tevian Dray and Corinne A. Manogue
World Scientific 2015
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`https://octonions.geometryof.org/GO`

Real Numbers

$$\mathbb{R}$$

Complex Numbers

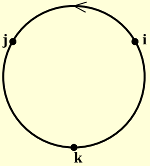
$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

Quaternions

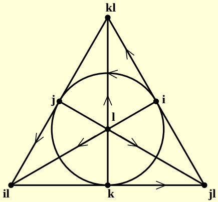
$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$



$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

Cayley–Dickson (1919)

Noncommutative:

$$ji = -ij$$

Nonassociative:

$$(ij)\ell = -i(j\ell)$$

Norm:

$$|x|^2 = x\overline{x}$$

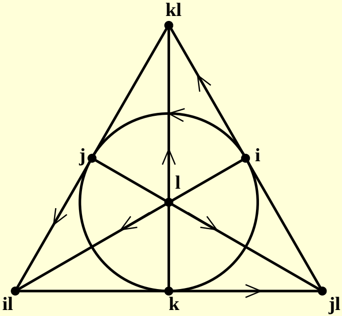
$$|x| = 0 \implies x = 0$$

Composition:

$$|xy| = |x||y|$$

Inverses (Division!):

$$x \neq 0 \implies x^{-1} = \overline{x}/|x|^2$$



The Standard Eigenvalue Problem

$$\boxed{\begin{array}{c} Av = \lambda v \\ (A^\dagger = A) \end{array}}$$

Reality: $\lambda \in \mathbb{R}$

Existence: $\exists n$ eigenvalues (counting multiplicity)

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow v_1^\dagger v_2 = 0$

Orthonormal Basis: $\exists$ orthonormal basis of eigenvectors

Decomposition: $A = \sum \lambda_m v_m v_m^\dagger$

"primitive idempotents"

Reality: $(\mathbf{A}^\dagger = \mathbf{A} \implies \bar{\lambda} = \lambda)$

$$Av = \lambda v \implies \bar{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v \neq \lambda v^\dagger v$$

$$Av = v\lambda \implies \bar{\lambda}(v^\dagger v) \neq (Av)^\dagger v \neq v^\dagger (Av) \neq (v^\dagger v)\lambda$$

Orthogonality: $(\lambda_1 \neq \lambda_2 \implies \mathbf{v}_1^\dagger \mathbf{v}_2 = 0)$

$$Av_m = \lambda_m v_m \implies \lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 \neq v_1^\dagger (Av_2) = \lambda_2 v_1^\dagger v_2$$

Example ($Av = \lambda v$ over $\mathbb{H}$ with $\lambda \notin \mathbb{R}$)

$$\mathbf{A}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1-k \\ 1+k \end{pmatrix} \implies \mathbf{A}_1 w_1 = j w_1$$

Example ($Av = v\lambda$ over $\mathbb{O}$ with $\lambda \notin \mathbb{R}$)

$$\mathbf{A}_2 = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} j \\ k\ell \end{pmatrix} \implies \mathbf{A}_2 w_2 = w_2 i$$

Characteristic Equation

$$\boxed{\mathcal{A} \in \mathbf{h}(3,\mathbb{O})}$$

$\Longrightarrow$

$$K[\mathcal{A}] = \mathcal{A}^3 - (\mathrm{tr}\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det\mathcal{A})\,\mathcal{I} = 0$$

with $\sigma(\mathcal{A}) = \frac{1}{2}\left((\mathrm{tr}\mathcal{A})^2 - \mathrm{tr}(\mathcal{A}^2)\right)$,
but

$$K[\lambda] = \lambda^3 - (\mathrm{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - (\det\mathcal{A}) = r_m$$

where $\{r_m, m = 1, 2\}$ are roots of a quadratic equation.

- Matrix solves characteristic equation;
- Eigenvalues do not;
- $\exists\, 2$ "families" of eigenvalues.

"Family" structure of $\mathbb{O}$

(Dray, Manogue, & Okubo 2002)

$$\mathbb{T} = \langle 1, a, b, c \rangle \subset \mathbb{O} \qquad \longleftrightarrow \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$$\Phi = \mathrm{Re}(a \times b \times c) = \tfrac{1}{2}\,\mathrm{Re}\big(a(\bar{b}c) - c(\bar{b}a)\big)$$

$$= \mathrm{Im}(a) \cdot [\mathrm{Im}(b) \times \mathrm{Im}(c)] \qquad \text{(triple product)}$$

$$\alpha = [a, b, c] = (ab)c - a(bc) \qquad \text{(associator)}$$

$$K[q] = r_m q \Longleftrightarrow q \in \mathbb{T}_m \subset \mathbb{O}; \quad r_m^2 - 4\Phi r_m - \alpha^2 = 0$$

$$\mathbb{T}_m = \mathbb{T} s_m; \quad s_m = \frac{r_m + 4\Phi + \alpha}{2(r_m + 2\Phi)}$$

$$\mathbb{O} = \mathbb{T}s_1 \oplus \mathbb{T}s_2 \qquad (s_1 + s_2 = 1)$$

$$\mathbb{T}_2 \equiv \mathbb{T}_1 \alpha \qquad (\mathbb{T}_1 \perp \mathbb{T}_2)$$

$$x, y \in \mathbb{T}_m \Longrightarrow x\bar{y} \in \mathbb{T}$$

The Right Eigenvalue Problem

$$\boxed{\begin{array}{c} Av = v\lambda \\ (A^\dagger = A) \end{array}}$$

Reality: Over $\mathbb{H}$, $\lambda \in \mathbb{R}$, but not over $\mathbb{O}$

Existence: 3×3 matrices over $\mathbb{O}$ have 2×3 real eigenvalues

Orthogonality: $\lambda_1 \neq \lambda_2 \Longrightarrow (v_1 v_1^\dagger) v_2 = 0$

Orthonormal Basis: $\exists$ 2 orthonormal bases of eigenvectors

Decomposition: $A = \sum \lambda_m (v_m v_m^\dagger)$ $(\times 2)$

The Jordan Eigenvalue Problem

(Dray & Manogue 1999)

$$\mathcal{A} \in \mathbf{h}(3, \mathbb{O})$$

$$\mathcal{V} \circ \mathcal{V} = \mathcal{V}$$

$$\mathcal{A} \circ \mathcal{B} = (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})/2$$

$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V}$$

Equivalent to right eigenvalue problem over $\mathbb{H}$! $(\mathcal{V} = vv^\dagger)$

$$(vv^\dagger) \circ (vv^\dagger) = (v^\dagger v)(vv^\dagger)$$

- usual characteristic equation
- $\lambda \in \mathbb{R}$
- Cayley–Moufang plane $(\mathbb{O}\mathbb{P}^2)$
- Solutions of 10-d Dirac equation!

Simultaneous Eigenstates

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$L_m\psi := -\frac{\hbar}{2}(\ell\sigma_m\psi)\ell$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow \qquad \begin{aligned} \mathbf{2\,L_z}\psi &= \quad \hbar\,\psi \\ \mathbf{2\,L_x}\psi &= -\hbar\,\psi\,\mathbf{k} \\ \mathbf{2\,L_y}\psi &= -\hbar\,\psi\,\mathbf{k}\boldsymbol{\ell} \end{aligned}$$

"spin-up" is simultaneous eigenstate of L_x, L_y, L_z!
(but **eigenvalues** don't commute!)

Dirac equation

Position space: $(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu})$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\hat{\Psi} = 0$$

Momentum space: $(\Psi = e^{-ip_\nu x^\nu/\hbar}\hat{\Psi})$

$$(\gamma^\mu p_\mu - mc)\Psi = 0$$

Weyl equation ($m = 0$): [up to normalization]

$$\mathbf{P}\Psi = 0 \Longrightarrow \mathbf{P} = \widetilde{\Psi\Psi^\dagger} = \Psi\Psi^\dagger - \Psi^\dagger\Psi \Longrightarrow \begin{pmatrix} \widetilde{\mathbf{P}} & \Psi \\ \Psi^\dagger & 1 \end{pmatrix} \in \mathbb{OP}^2$$

Works in 3,4,6,10 spacetime dimensions! Supersymmetry!!

Octonionic projections are quaternionic!

$(a, b, c \in \mathbb{O};\ x, y, z \in \mathbb{R})$

$$\mathcal{A} = \begin{pmatrix} x & a & \bar{c} \\ \bar{a} & y & b \\ c & \bar{b} & z \end{pmatrix}$$

$$\mathcal{A}^2 = \begin{pmatrix} x^2 + |a|^2 + |c|^2 & (x+y)a + \bar{c}\bar{b} & (x+z)\bar{c} + ab \\ (x+y)\bar{a} + bc & |a|^2 + y^2 + |b|^2 & (y+z)b + \overline{ac} \\ (x+z)c + \bar{b}\bar{a} & (y+z)\bar{b} + ca & |c|^2 + |b|^2 + z^2 \end{pmatrix}$$

$$\mathcal{A} \in \mathbb{OP}^2 \Longrightarrow \mathcal{A}^2 = \mathcal{A} \Longrightarrow ab = (1 - x - z)\,\bar{c} \Longrightarrow [a, b, c] = 0!$$

Application: Solutions to 10-d Dirac equation (octonionic) are in fact 6-d (quaternionic), leaving room for additional symmetry.

Albert Algebra ($\mathbf{h}(3,\mathbb{O})$)

- *Only* nonassociative Jordan algebra (over $\mathbb{R}$)
- Also referred to as the **exceptional Jordan algebra**.

Jordan product:

$$\mathcal{A} \circ \mathcal{B} = \frac{1}{2} (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})$$

$$u, v, w \in \mathbb{R}^3 \Longrightarrow \mathrm{tr}\big[(vv^\dagger) \circ (ww^\dagger)\big] = (v \cdot w)^2$$

Freudenthal product:

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})\,\mathcal{Y} + (\mathrm{tr}\mathcal{Y})\,\mathcal{X}\Big) + \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})(\mathrm{tr}\mathcal{Y}) - \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I}$$

$$(vv^\dagger) * (ww^\dagger) = \tfrac{1}{2}(v \times w)(w \times w)^\dagger$$

Determinant:

$$\det \mathcal{X} = \frac{1}{3}\mathrm{tr}\,(\mathcal{A} \circ (\mathcal{A} * \mathcal{A}))$$

$$\mathrm{tr}\big[(uu^\dagger) \circ \big((vv^\dagger) * (ww^\dagger)\big)\big] = \tfrac{1}{2}\big(u \cdot (v \times w)\big)^2$$

Projective plane:

$$\mathcal{A}^2 = \mathcal{A},\ \ \mathrm{tr}\mathcal{A} = 1 \Longleftrightarrow \mathcal{A} * \mathcal{A} = 0$$

The Tits–Freudenthal Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):
$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_{6(-26)} \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{Spin}(9,1)$
Dray, Manogue, & Wilson (2014): $E_7 \cong \mathrm{Sp}(6,\mathbb{O})$
Wilson, Dray, & Manogue (2023): $E_8 \cong \mathrm{SU}(3,\mathbb{O}'\otimes\mathbb{O})$
Manogue, Dray, & Wilson (2022): Standard Model from $\mathfrak{e}_8$

Minimal representation of $\mathfrak{e}_8$ is adjoint!

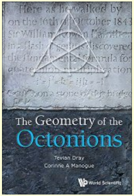
The algebras in the 3 × 3 magic square are $\mathfrak{su}(3,\mathbb{K}'\otimes\mathbb{K})$.

SUMMARY

- Real eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Always get decompositions into primitive idempotents;
- Splits octonions into two "almost quaternionic" subspaces!
- Jordan eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Primitive idempotents are quaternionic! ($\mathbb{O}\mathbb{P}^2$)
- Applications to physics: spin, Dirac equation...

> *Only wimps specialize in the general case; real scientists pursue examples.*
>
> (attributed to Beresford Parlett by Sir Michael Berry)

$$\mathbb{O} \longmapsto \mathbf{h}_3(\mathbb{O}) \longmapsto \mathfrak{e}_8 \longmapsto \text{nature?}$$



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