

The Octonionic Eigenvalue Problem

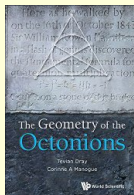
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Book



The Geometry of the Octonions

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World Scientific 2015

ISBN: 978-981-4401-81-4

<https://octonions.geometryof.org/G0>



Real Numbers

$$\mathbb{R}$$

Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

$$i^2 = j^2 = \ell^2 = -1$$

Cayley–Dickson (1919)

(Real and) Complex Numbers

$$z = a + bi \in \mathbb{C}$$

$$\bar{z} = a - bi \quad |z|^2 = z\bar{z}$$

Commutative:

$$z_2 z_1 = z_1 z_2$$

Associative:

$$z_1(z_2 z_3) = (z_1 z_2)z_3$$

Composition:

$$|z_1 z_2| = |z_1| |z_2|$$

Inverse:

$$z^{-1} = \frac{\bar{z}}{|z|^2} \quad (z \neq 0)$$

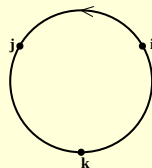
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

$$i^2 = j^2 = \ell^2 = -1$$

Cayley–Dickson (1919)

Quaternions

$$q = a + bi + cj + dk \in \mathbb{H}$$
$$\bar{q} = a - bi - cj - dk \quad |q|^2 = q\bar{q}$$

Noncommutative: $ji = -ij$

Associative: $q_1(q_2q_3) = (q_1q_2)q_3$

Composition: $|q_1q_2| = |q_1||q_2|$

Inverse: $q^{-1} = \frac{\bar{q}}{|q|^2} \quad (q \neq 0)$

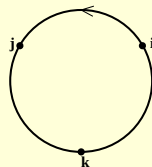
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



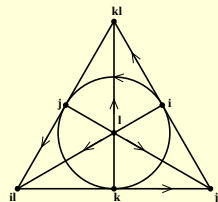
Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$



$$i^2 = j^2 = \ell^2 = -1$$

Cayley–Dickson (1919)

Noncommutative:

$$ji = -ij$$

Nonassociative:

$$(ij)\ell = -i(j\ell)$$

Norm:

$$|x|^2 = x\bar{x}$$

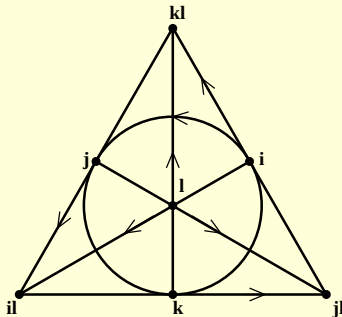
$$|x| = 0 \implies x = 0$$

Composition:

$$|xy| = |x||y|$$

Inverses (Division!):

$$x \neq 0 \implies x^{-1} = \bar{x}/|x|^2$$



Division Algebras

Theorem (Hurwitz 1923)

The only normed division algebras over the reals are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.

Hermitian Matrices

Matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Transpose:

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Conjugate:

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

Dagger:

$$A^\dagger = \bar{A}^T = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

Hermitian:
($p, m \in \mathbb{R}$)

$$A^\dagger = A \implies A = \begin{pmatrix} p & b \\ \bar{b} & m \end{pmatrix}$$

(over $\mathbb{R}$, Hermitian $\equiv$ symmetric)

The Standard Eigenvalue Problem

$$\begin{aligned} Av &= \lambda v \\ (A^\dagger &= A) \end{aligned}$$

Reality: $\lambda \in \mathbb{R}$

Existence: $\exists n$ eigenvalues (counting multiplicity)

Orthogonality: $\lambda_1 \neq \lambda_2 \implies v_1^\dagger v_2 = 0$

Orthonormal Basis: $\exists$ orthonormal basis of eigenvectors

Decomposition: $A = \sum \lambda_m v_m v_m^\dagger$
“primitive idempotents”

Reality: $(A^\dagger = A \implies \bar{\lambda} = \lambda)$

$$Av = \lambda v \implies \bar{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger Av = v^\dagger \lambda v \neq \lambda v^\dagger v$$

$$Av = v\lambda \implies \bar{\lambda}(v^\dagger v) \neq (Av)^\dagger v \neq v^\dagger(Av) \neq (v^\dagger v)\lambda$$

Orthogonality: $(\lambda_1 \neq \lambda_2 \implies v_1^\dagger v_2 = 0)$

$$Av_m = \lambda_m v_m \implies \lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 \neq v_1^\dagger(Av_2) = \lambda_2 v_1^\dagger v_2$$

Example $(Av = \lambda v \text{ over } \mathbb{H} \text{ with } \lambda \notin \mathbb{R})$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1-k \\ 1+k \end{pmatrix} \implies \mathbf{A}_1 w_1 = jw_1$$

Example $(Av = v\lambda \text{ over } \mathbb{O} \text{ with } \lambda \notin \mathbb{R})$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} j \\ k\ell \end{pmatrix} \implies \mathbf{A}_2 w_2 = w_2 i$$

Characteristic Equation

$$\mathcal{A} \in \mathfrak{h}(3, \mathbb{O})$$

$\Rightarrow$

$$K[\mathcal{A}] = \mathcal{A}^3 - (\text{tr} \mathcal{A}) \mathcal{A}^2 + \sigma(\mathcal{A}) \mathcal{A} - (\det \mathcal{A}) \mathcal{I} = 0$$

with $\sigma(\mathcal{A}) = \frac{1}{2} ((\text{tr} \mathcal{A})^2 - \text{tr}(\mathcal{A}^2))$,

but

$$K[\lambda] = \lambda^3 - (\text{tr} \mathcal{A}) \lambda^2 + \sigma(\mathcal{A}) \lambda - (\det \mathcal{A}) = r_m$$

where $\{r_m, m = 1, 2\}$ are roots of a quadratic equation.

- Matrix solves characteristic equation;
- Eigenvalues do not;
- $\exists 2$ “families” of eigenvalues.

The Right Eigenvalue Problem

$$\begin{aligned} Av &= v\lambda \\ (A^\dagger &= A) \end{aligned}$$

Reality: Over $\mathbb{H}$, $\lambda \in \mathbb{R}$, but not over $\mathbb{O}$

Existence: 3×3 matrices over $\mathbb{O}$ have 2×3 real eigenvalues

Orthogonality: $\lambda_1 \neq \lambda_2 \implies (v_1 v_1^\dagger) v_2 = 0$

Orthonormal Basis: $\exists$ 2 orthonormal bases of eigenvectors

Decomposition: $A = \sum \lambda_m (v_m v_m^\dagger)$ ($\times 2$)

Proof: 8-hour brute force computation using Mathematica
(Dray & Manogue 1998)

The Jordan Eigenvalue Problem

(Dray & Manogue 1999)

$$\mathcal{A} \in \mathfrak{h}(3, \mathbb{O})$$

$$\mathcal{V} \circ \mathcal{V} = \mathcal{V}$$

$$\mathcal{A} \circ \mathcal{B} = (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})/2$$

$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V}$$

Equivalent to right eigenvalue problem over $\mathbb{H}$! ($\mathcal{V} = v v^\dagger$)

$$(v v^\dagger) \circ (v v^\dagger) = (v^\dagger v)(v v^\dagger)$$

- usual characteristic equation
- $\lambda \in \mathbb{R}$
- Cayley–Moufang plane ($\mathbb{O}\mathbb{P}^2$)
- Solutions of 10-d Dirac equation!

Simultaneous Eigenstates

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$L_m \psi := -\frac{\hbar}{2} (\ell \sigma_m \psi) \ell$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \implies \begin{aligned} 2 \mathbf{L}_z \psi &= \hbar \psi \\ 2 \mathbf{L}_x \psi &= -\hbar \psi \mathbf{k} \\ 2 \mathbf{L}_y \psi &= -\hbar \psi \mathbf{k} \ell \end{aligned}$$

“spin-up” is simultaneous eigenstate of L_x, L_y, L_z !
(but **eigenvalues** don't commute!)

Dirac equation

Position space: $(\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu})$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\hat{\Psi} = 0$$

Momentum space: $(\Psi = e^{-ip_\nu x^\nu/\hbar}\hat{\Psi})$

$$(\gamma^\mu p_\mu - mc)\Psi = 0$$

Weyl equation ($m = 0$): [up to normalization]

$$\mathbf{P}\Psi = 0 \implies \mathbf{P} = \widetilde{\Psi\Psi^\dagger} = \Psi\Psi^\dagger - \Psi^\dagger\Psi \implies \begin{pmatrix} \tilde{\mathbf{P}} & \Psi \\ \Psi^\dagger & 1 \end{pmatrix} \in \mathbb{OP}^2$$

Works in 3,4,6,10 spacetime dimensions! Supersymmetry!!

Simple Lie Groups

Theorem (Killing 1888–1890; E. Cartan 1894)

The only simple Lie groups are the orthogonal matrices (over $\mathbb{R}$), unitary matrices (over $\mathbb{C}$), and symplectic matrices (over $\mathbb{H}$) of any size, and five exceptional cases involving the octonions ($\mathbb{O}$).

The Tits–Freudenthal Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3, \mathbb{R})$	$\mathfrak{su}(3, \mathbb{C})$	$\mathfrak{su}(3, \mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{su}(3, 3, \mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3, \mathbb{O})$, $E_{6(-26)} \cong \mathrm{SL}(3, \mathbb{O})$ using $\mathrm{SL}(2, \mathbb{O}) \cong \mathrm{Spin}(9, 1)$

Dray, Manogue, & Wilson (2014): $E_7 \cong \mathrm{Sp}(6, \mathbb{O})$

Wilson, Dray, & Manogue (2023): $E_8 \cong \mathrm{SU}(3, \mathbb{O}' \otimes \mathbb{O})$

Manogue, Dray, & Wilson (2022): Standard Model from $\mathfrak{e}_8$

Minimal representation of $\mathfrak{e}_8$ is adjoint!

The algebras in the 3×3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.

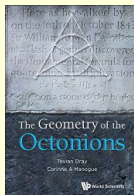
SUMMARY

- Real eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Always get decompositions into primitive idempotents;
- Jordan eigenvalue problem over $\mathbf{h}(3, \mathbb{O})$ well understood;
- Applications to physics: spin, Dirac equation...

Only wimps specialize in the general case; real scientists pursue examples.

(attributed to Beresford Parlett by Sir Michael Berry)

$$\mathbb{O} \longmapsto \mathbf{h}_3(\mathbb{O}) \longmapsto \mathfrak{e}_8 \longmapsto \text{nature?}$$



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