Putting Differentials back into Calculus

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Infinitesimals

"ghosts of departed quantities" (George Berkeley)

People used to believe in it, and now they have found out their mistake. (Bertand Russell)

... many mathematicians think in terms of infinitesimal quantities: apparently, however, real mathematicians would never allow themselves to write down such thinking, at least not in front of the children.
(Bill McCallum)

Infinitesimals

I think, in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus.

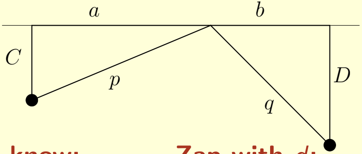
(Abraham Robinson)

But so great is the average person's fear of the infinite that to this day calculus all over the world is being taught as a study of limit processes instead of what it really is: infinitesimal analysis. *(Rudy Rucker)*

Just go on ... and faith will soon return.

(d'Alembert)

Example: Where to pump water from a river



Use what you know:

$$\begin{aligned} a+b &= S \\ a^2+C^2 &= p^2 \\ b^2+D^2 &= q^2 \\ p+q &= \ell \end{aligned}$$

Zap with d:

$$\begin{aligned} da+db &= 0 \\ 2a\,da &= 2p\,dp \\ 2b\,db &= 2q\,dq \\ dp+dq &= d\ell \end{aligned}$$

$$0 = d\ell = dp + dq = \frac{a}{p}\,da + \frac{b}{q}\,db = \left(\frac{a}{p} - \frac{b}{q}\right) da$$

$$\Longrightarrow \frac{b^2}{a^2} = \frac{q^2}{p^2} = \frac{b^2+D^2}{a^2+C^2} \Longrightarrow \frac{b}{a} = \frac{D}{C}$$

$$a = \frac{CS}{C+D} \qquad b = \frac{DS}{C+D}$$

Differentials

$$
\begin{aligned}
d(u + cv) &= du + c\,dv \\
d(uv) &= u\,dv + v\,du
\end{aligned}
$$

$$
\begin{aligned}
d\left(u^n\right) &= nu^{n-1}\,du \\
d\left(e^u\right) &= e^u\,du \\
d(\sin u) &= \cos u\,du \\
d(\cos u) &= -\sin u\,du \\
d(\ln u) &= \frac{1}{u}\,du
\end{aligned}
$$

Derivatives (& Integrals)

Derivatives:

$$\frac{d}{du}\sin u = \frac{d\sin u}{du} = \frac{\cos u\,du}{du} = \cos u$$

Chain rule:

$$\frac{d}{dx}\sin u = \frac{d\sin u}{dx} = \frac{\cos u\,du}{dx} = \cos u\,\frac{du}{dx}$$

Inverse functions:

$$q = \ln u \Longrightarrow e^q = u \Longrightarrow d(e^q) = e^q\,dq = du \Longrightarrow dq = \frac{du}{e^q} = \frac{du}{u}$$

Integrals:

$$\int 2x\,cos(x^2)\,dx = \int \cos u\,du = \int d(\sin u) = \sin u = \sin(x^2)$$

Derivatives

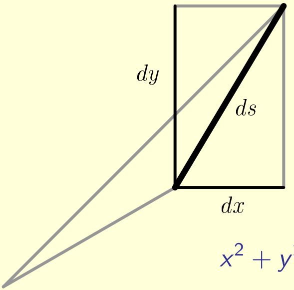
Instead of:

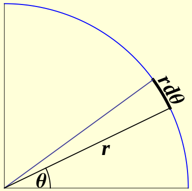
- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with d"

Trig Differentials





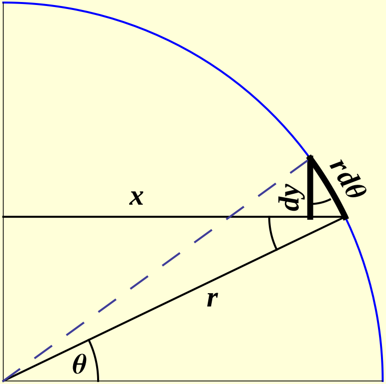
$$x^2 + y^2 = r^2 \Longrightarrow x\,dx + y\,dy = 0$$

$$ds^2 = r^2\,d\theta^2 = dx^2 + dy^2 = dx^2\left(1 + \frac{x^2}{y^2}\right) = r^2\,\frac{dx^2}{y^2},$$

$$d\theta^2 = \frac{dx^2}{y^2} = \frac{dy^2}{x^2} \qquad \Longrightarrow \qquad \begin{aligned} dy &= x\,d\theta \\ dx &= -y\,d\theta \end{aligned}$$

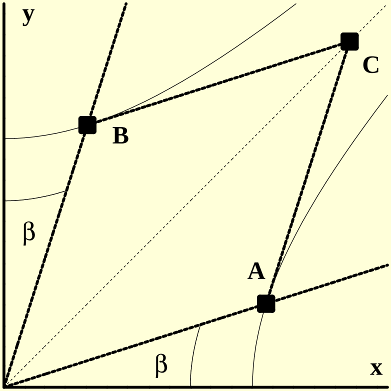
$$\Longrightarrow \qquad \boxed{\begin{aligned} d\sin\theta &= \cos\theta\,d\theta \\ d\cos\theta &= -\sin\theta\,d\theta \end{aligned}}$$

Proof Without Words



$$dy = (r\,d\theta)\cos\theta$$
$$d(r\sin\theta) = r\,\cos\theta\,d\theta$$
$$d(\sin\theta) = \cos\theta\,d\theta$$

Exponential Function

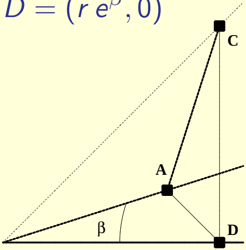


$A = (r \cosh \beta, r \sinh \beta)$

$B = (r \sinh \beta, r \cosh \beta)$

$C = A + B = (r\, e^{\beta}, r\, e^{\beta})$

$D = (r\, e^{\beta}, 0)$



in Minkowski space...

Differentials of Functions

Linear approximation:

$$y = f(x) \Longrightarrow \Delta y \approx f'(x)\,\Delta x$$

“Modern” differential:

RHS *defined* to be dy

Just another (finite) variable:

$$F(x, dx) = f(x) + f'(x)\,dx$$

- $dx = \Delta x$, but $dy \neq \Delta y$
- dx finite

Differentials of Variables

$$df = \frac{df}{dx}\,dx$$

- Shorthand for limit argument
- Nonstandard analysis (hyperreal numbers)
- Smooth infinitesimal analysis
- Differential forms

Summary

- Calculus is not about functions.
- Calculus is not about limits.
- Science is about the relationship between physical quantities.
- Calculus is about small changes in those relationships.

- Calculus was used successfully before Cauchy et al.
- Differentials used by many scientists and engineers.
- Differentials correctly reproduce rules of calculus

PUT DIFFERENTIALS BACK
into
Differential (and Integral)
CALCULUS

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