Geometric Reasoning in Multivariable Calculus

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What are Functions?

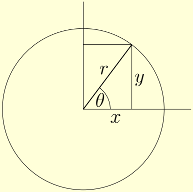
Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$



What are Functions?

MATH

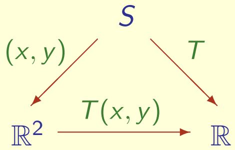
$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

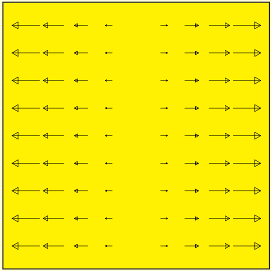
Differential Geometry!

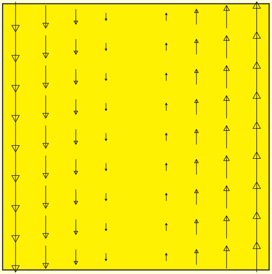
$$T(x,y) \longleftrightarrow T \circ (x,y)^{-1}$$
$$T(r,\theta) \longleftrightarrow T \circ (r,\theta)^{-1}$$



Mathematics and Physics are two disciplines separated by a common language!

Geometric Reasoning





- Which vector field is conservative?
- Which vector field has nonzero curl?
- Which vector field has nonzero divergence?

Which vector field could represent a (static) electric field? a (static) magnetic field?

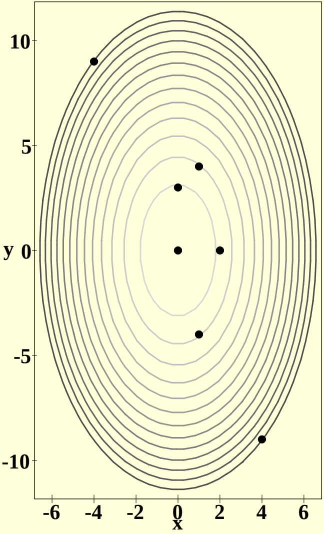
$$(\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\Phi \Longrightarrow \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0; \quad \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \Longrightarrow \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0)$$

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

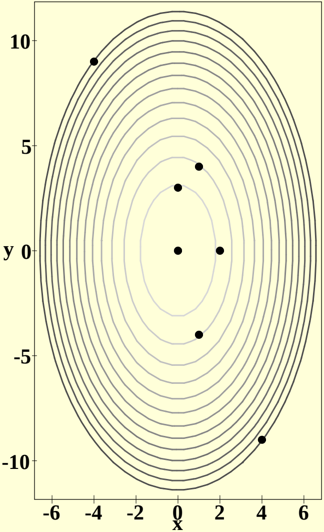
where $a = 5000$ ft, $b = 30 \,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10 \,\frac{\text{ft}}{\text{mi}^2}$.



Kinesthetic Activity

Stand up and close your eyes.

Hold out your right arm in the direction of the gradient where you are standing.



Surfaces

(Each surface is dry-erasable, as are the matching contour maps.)

Raising Calculus to the Surface (Aaron Wangberg)

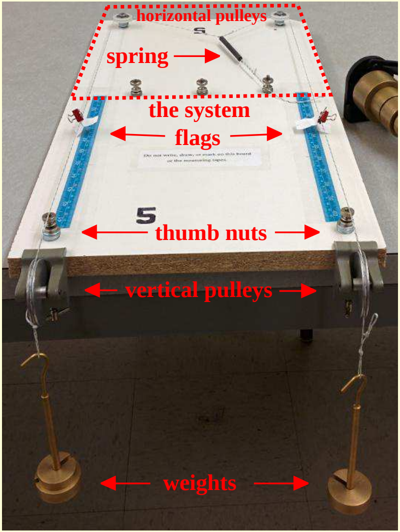
Raising Physics to the Surface (+ Liz Gire, Robyn Wangberg)

https://raisingcalculus.winona.edu

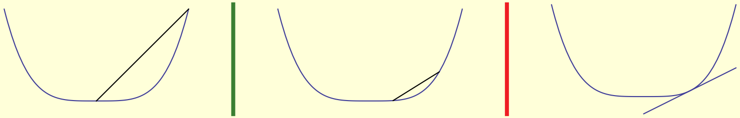
Partial Derivative Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.
- Mathematicians:
  "That's not a derivative!"

Roundy et al., *Experts' Understanding of Partial Derivatives Using the Partial Derivative Machine*, PERC 2014



Thick Derivatives



Math: ∃ "bright line" between *average* rate of change and *instantaneous* rate of change.

(Such averages are used to approximate derivatives.)

Physics: "Average" refers to secant lines, not (good) approximations to tangent lines.

Move the bright line!

Thick Derivatives!

(Derivatives are fundamentally ratios of small changes, not limits.)

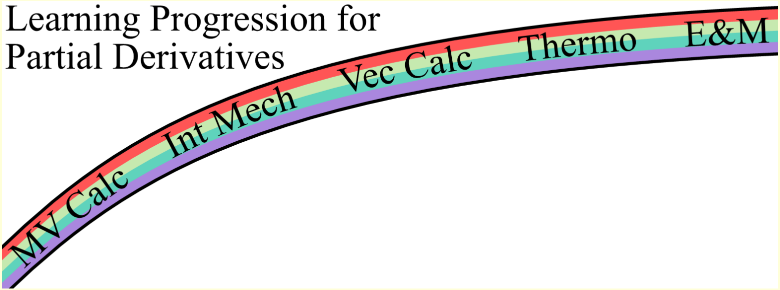
[Dray, AMS Blog on Education, 5/31/16]

The Paradigms in Physics Project

- Complete redesign of physics major – 20 new courses
- Junior-year “paradigms” designed around common themes.
- Senior-year “capstones” finish traditional disciplinary content.
- 25+ years of continuous NSF funding.
- Living curriculum: *Monthly curriculum meetings for 25+ years!*
- Paradigms 2.0 implemented in 2017.
- Active engagement: *300+ documented activities!*

https://paradigms.oregonstate.edu

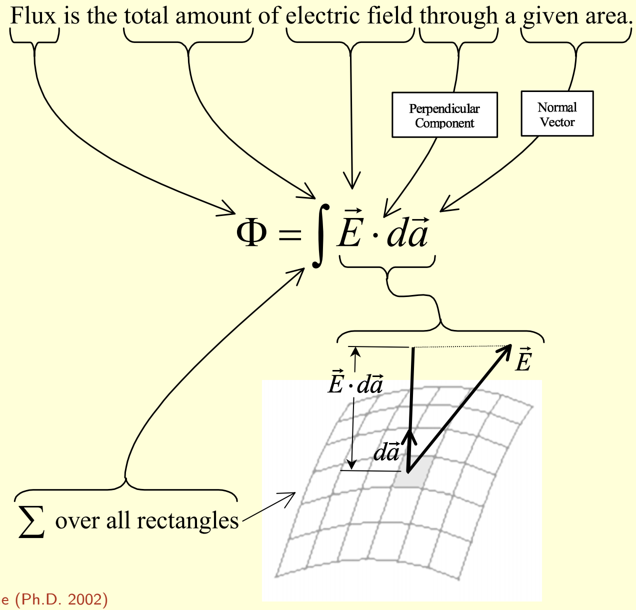
Learning Progression



- Successively more sophisticated ways of thinking about a topic.
- Sequences supported by research on learner's ideas and skills.
- *Lower anchor* grounded in students' prior ideas and skills.
- *Upper anchor* grounded in knowledge and practices of experts.

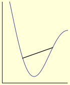
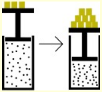
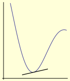
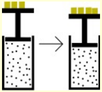
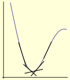
Duschle et al., NRC, 2007; Plummer, 2012; Sikorski et al., 2009, 2010
Manogue, Dray, Emigh, Gire, & Roundy, PERC 2017

Multiple Representations



Kerry Browne (Ph.D. 2002)

Extended Theoretical Framework for Concept of Derivative

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"avg. rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ numerically	
Limit		"inst. rate of change"	$\lim\limits_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"...at any point/time"	$f'(x) =$ ...	... depends on x	tedious repeti- tion

No entry for symbolic differentiation!!

Roundy, Dray, Manogue, Wagner, & Weber, CRUME 18 Proceedings, MAA, 2015. `https://sigmaa.maa.org/rume/Site/Proceedings.html`

Differentials

Does $\frac{df}{dx}$ mean "$f'(x)$" or "df over dx"?

$$d(u^2) = 2u\,du$$
$$d(\sin u) = \cos u\,du$$

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

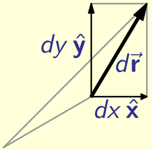
One coherent idea:

"Zap equations with *d*"

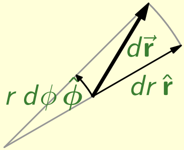
(infinitesimal reasoning)

Dray & Manogue, CMJ **34**, 283–290 (2003); CMJ **41**, 90–100 (2010).

Vector calculus is about one coherent concept:
Infinitesimal Displacement



$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$

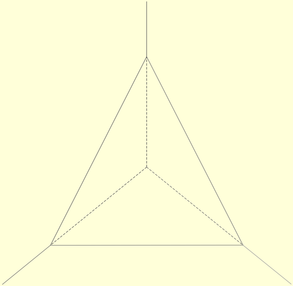


$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$$

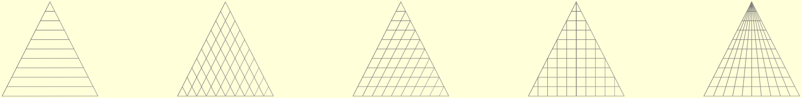
$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

Flux

What is the flux of the vector field $\vec{\mathbf{E}} = z\,\hat{\mathbf{z}}$ upwards through the triangular region connecting the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$?



First decide how to chop up the region:



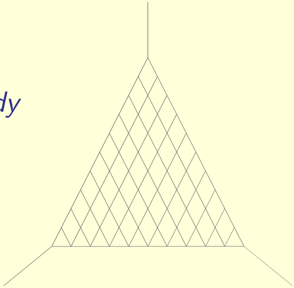
Use what you know!

Chop parallel to the x and y axes:

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$\{x + y + z = 1\} \Longrightarrow$

$$\begin{aligned} d\vec{\mathbf{r}}_1 &= (\,\hat{\mathbf{x}} - \hat{\mathbf{y}}\,)\,dx \qquad (y = \text{const}) \\ d\vec{\mathbf{r}}_2 &= (\,\hat{\mathbf{y}} - \hat{\mathbf{z}}\,)\,dy \qquad (x = \text{const}) \\ \Longrightarrow d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 = (\,\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}\,)\,dx\,dy \end{aligned}$$



$\vec{\mathbf{E}} = z\,\hat{\mathbf{z}} \Longrightarrow$

$$\int_T \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_0^1 \int_0^{1-y} (1-x-y)\,dx\,dy = \frac{1}{6}$$

CUPM

MAA Committee on the Undergraduate Program in Mathematics

Curriculum Guide

https://www.maa.org/cupm/cupm2004.pdf

CRAFTY

Subcommittee on Curriculum Renewal Across the First Two Years

Voices of the Partner Disciplines

https://www.maa.org/cupm/crafty

SUMMIT-P

https://www.summit-p.com

SUMMARY

- Use multiple representations, including geometry, measurement, numerical data;
- Always ask both "With respect to what," **and** "With what held constant."

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https://math.oregonstate.edu/bridge
https://books.physics.oregonstate.edu/GVC
https://paradigms.oregonstate.edu
https://raisingcalculus.winona.edu