Octions:
An E_8 Description of the Standard Model

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References

This work:

An octonionic construction of E8 and the Lie algebra magic square, Wilson, Dray, & Manogue, Innov. Incidence Geom. (to appear), `arXiv:2204.04996`.

Octions: An E8 description of the Standard Model, Manogue, Dray, & Wilson, J. Math. Phys. **63**, 081703 (2022), `arXiv:2204.05310`.

Our group: Fairlie & Manogue (1986, 1987), Manogue & Sudbery (1989), Schray (PhD 1994), Manogue & Schray (1993), Dray & Manogue (1998ab, 1999), Manogue & Dray (1999), Dray, Janesky, & Manogue (2000), Dray, Manogue, & Okubo (2002), Dray & Manogue (CAA 2000, CMUC 2010), Manogue & Dray (2010), Wangberg (PhD 2007), Wangberg & Dray (JMP 2013, JAA 2014), Dray, Manogue, & Wilson (CMUC 2014), Kincaid (MS 2012), Kincaid and Dray (MPLA 2014), Dray, Huerta, & Kincaid (LMP 2014)

Others: Jordan (1933), Jordan, von Neumann, & Wigner (1934), Freudenthal (1954, 1964), Tits (1966), Vinberg (1966), Gürsey, Ramond, & Sikivie (1976), Olive & West (1983), Kugo & Townsend (1983), Günaydin & Gürsey (1987), Chung & Sudbery (1987), Goddard, Nahm, Olive & Ruegg (1987), Corrigan & Hollowood (1988), Dixon (1994), Okubo (1995), Günaydin, Koepsell, & Nicolai (2001), Barton & Sudbery (2003), Cederwall (2007), Lisi (2007, 2010), Baez & Huerta (2010), Chester, Marran, & Rios (2021), Furey (2015), Furey & Hughes (2022ab)

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$sa(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Description as symmetry groups

[Barton & Sudbery (2003), Wangberg (PhD 2007),
Dray & Manogue (CMUC 2010), Wangberg & Dray (JMP 2013, JAA 2014),
Dray, Manogue, & Wilson (CMUC 2014), Wilson, Dray, & Manogue (2022)]

Guiding Principle #1

Lie algebras are real!

(signature matters)

$\mathfrak{so}(3,1)$ has boosts and rotations

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Guiding Principle #2

The 3×3 structure is broken to 2×2.

$$\mathcal{P} = \begin{pmatrix} P & \theta \\ -\theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P} \longmapsto \mathcal{M}\mathcal{P}\mathcal{M}^{-1} \quad \Longrightarrow \quad P \longmapsto MPM^{-1},\ \theta \longmapsto M\theta$$

$$\mathcal{P} \longmapsto [\mathcal{A}, \mathcal{P}] \quad \Longrightarrow \quad P \longmapsto [A, P],\ \theta \longmapsto A\theta$$

Idea: Adjoint and spinor actions at same time!

Example: $\mathcal{M} \in E_6$, $\mathcal{A} \in \mathfrak{e}_6, \mathcal{P} \in \mathfrak{e}_6$

Summary: 2×2 Magic Square

- The algebras in the 2×2 magic square are $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 2×2 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad X_p = \begin{pmatrix} 0 & p \\ -\overline{p} & 0 \end{pmatrix}$$

Idea: rotations/boosts acting on $\mathbb{K}' \oplus \mathbb{K}$:

$$D_i = D_{1i};\ D_L = D_{UL};\ X_i = X_{iU};\ X_L = X_{1L}$$

- The remaining basis elements are of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 \\ 0 & i \circ j \end{pmatrix} = \frac{1}{2} [D_i, D_j]$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$.

Summary: 3×3 Magic Square

- The algebras in the 3×3 magic square are $\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$.
- Each algebra is generated by the 3×3 matrices below, with $p \in \mathbb{K}' \otimes \mathbb{K}$ and $q \in \mathrm{Im}\mathbb{K} + \mathrm{Im}\mathbb{K}'$.

$$D_q = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}, \quad X_p = \begin{pmatrix} 0 & p & 0 \\ -\overline{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\overline{p} & 0 \end{pmatrix}, \quad Z_p = \begin{pmatrix} 0 & 0 & -\overline{p} \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}$$

- The remaining basis elements can be chosen to be of the form

$$D_{i,j} = \begin{pmatrix} i \circ j & 0 & 0 \\ 0 & i \circ j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $i \circ j \doteq k$ over $\mathbb{H}$, but stands for nesting over $\mathbb{O}$. **TRIALITY!**

Magic Squares

2 × 2 "half-split": (spin groups ...)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

3 × 3 "half-split": (... plus spinor reps)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Subalgebras

- All algebras in both magic squares are subalgebras of $\mathfrak{e}_8$!
- $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \mathbf{128}$.
- The $\mathbf{128}$ is a Majorana–Weyl representation of $\mathfrak{so}(12,4)$.
- The $\mathbf{128}$ contains spinor reps of each 2×2 algebra.

Guiding Principle #3

All representations live in $\mathfrak{e}_8$!

$$\begin{aligned}
\mathfrak{e}_{8(-24)} &= \mathfrak{so}(12,4) + \text{spinors} \\
\mathfrak{so}(12,4) &\supset \mathfrak{so}(3,1) + \mathfrak{so}(7,3) + \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) + \mathfrak{so}(4) + \mathfrak{so}(3,3) + \mathfrak{so}(2) \\
&\supset \mathfrak{so}(3,1) + \mathfrak{su}(2)_L + \mathfrak{su}(2)_R + \mathfrak{su}(3)_c + \mathfrak{u}(1) + \mathfrak{so}(2)
\end{aligned}$$

Decomposition of $\mathfrak{so}(12,4)$

$$\mathfrak{e}_{8(-24)} \supset \mathfrak{so}(12,4) + \overbrace{\text{spinors}}^{\text{complex}}$$

$$\mathfrak{so}(12,4) \supset \underbrace{\mathfrak{so}(2)}_{\substack{\text{complex}\\ \text{structure}}} + \mathfrak{so}(3,1) + \underbrace{\mathfrak{so}(7,3)}_{\text{“}\mathfrak{so}(10)\text{”}}$$

Lorentz + $\mathfrak{so}(10)$ GUT

Spinors are complex

Color Decomposition of $\mathfrak{e}_8$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

$$\mathfrak{e}_{8(-24)} = \begin{cases} \cancel{\mathfrak{e}_{6(2)} + 54 \times 3 + \overbrace{\mathfrak{su}(3)}^{\text{compact}}} \\ \mathfrak{e}_{6(-26)} + \underbrace{27}_{\substack{\text{Jordan}\\ \text{algebra}}} \times 6 + \mathfrak{sl}(3,\mathbb{R}) \end{cases}$$

Decomposition of $\mathfrak{e}_8 \longrightarrow \mathfrak{e}_6$

$$\mathfrak{so}(12,4) = \mathfrak{so}(9,1) + 10 \times 6 + \overbrace{\mathfrak{so}(3,3)}^{\mathfrak{sl}(3,\mathbb{R}) + 3 \times 2 + \langle S_L \rangle}$$

Majorana–Weyl spinors of both $\mathfrak{so}(9,1)$ & $\mathfrak{so}(3,3)$

$$128 = 16 \times (1 + 1 + 3 + \overline{3})$$

$$\mathfrak{e}_{8(-24)} = \underbrace{\mathfrak{e}_{6(-26)}}_{} + \underbrace{27 \times 6}_{} + \mathfrak{sl}(3,\mathbb{R})$$

$\mathfrak{so}(9,1)$ $16 \times 2 \times 1$ S_L	$10 \times 2 \times 3$ $16 \times 2 \times 3$ $1 \times 2 \times 3$

Spinors of $\mathfrak{e}_6$: Volume Elements

$$\mathfrak{e}_{6(-26)} = \mathfrak{so}(9,1) + \overbrace{16 \times 2}^{\text{Killing duals}} + \langle S_L \rangle$$

$$\mathfrak{so}(9,1) \longmapsto \mathfrak{so}(2) + \mathfrak{so}(7,1) \longmapsto \mathfrak{so}(2) + \overbrace{\mathfrak{so}(3,1)}^{\text{Lorentz}} + \underbrace{\overbrace{\mathfrak{so}(4)}^{\text{weak}}}_{\mathfrak{su}(2)_L + \mathfrak{su}(2)_R}$$

SUMMARY

Everything in $\mathfrak{e}_{8(-24)}$!

$$\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) + \text{spinors}$$

$$= \underbrace{\mathfrak{e}_{6(-26)}}_{\mathfrak{so}(9,1) + 2\times 16 + \langle S_L \rangle} + 27\times 6 + \mathfrak{sl}(3,\mathbb{R})$$

gluons & quarks

$$\underbrace{\mathfrak{so}(9,1)} + 2\times 16 + \langle S_L \rangle$$

leptons

$$\mathfrak{so}(2) + \mathfrak{so}(3,1) + \underbrace{\mathfrak{su}(2)_L + \mathfrak{su}(2)_R}_{\text{3 generations}} + \underbrace{4\times 4}_{\text{electroweak mediators}}$$

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