The Geometry of Calculus

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Acknowledgments

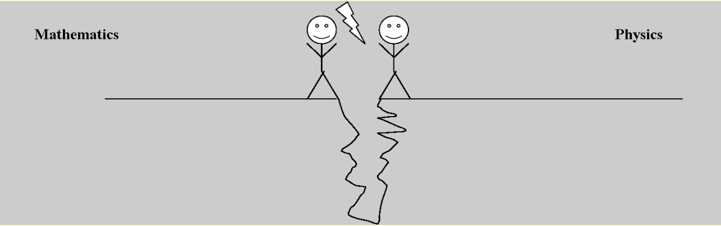
Places:

People:

Stuart Boersma, Johnner Barrett, Sam Cook, Aaron Wangberg

Corinne Manogue

Mathematics vs. Physics



- **Physics is about things.**
- **Physicists can't change the problem.**

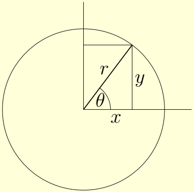
- **Mathematicians do algebra.**
- **Physicists do geometry.**

What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x,y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r,\theta)$?



A: $T(r,\theta) = kr^2$

B: $T(r,\theta) = k(r^2 + \theta^2)$

What are Functions?

MATH

$$\begin{aligned} T &= f(x,y) = k(x^2+y^2) \\ T &= g(r,\theta) = kr^2 \end{aligned}$$

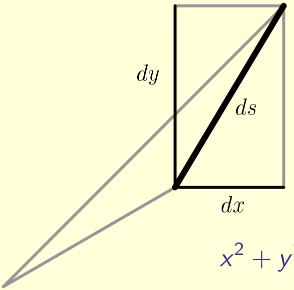
PHYSICS

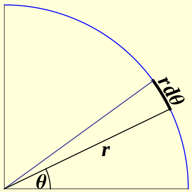
$$\begin{aligned} T &= T(x,y) = k(x^2+y^2) \\ T &= T(r,\theta) = kr^2 \end{aligned}$$

Two disciplines separated by a common language...

physical quantities $\neq$ functions

Trig Differentials





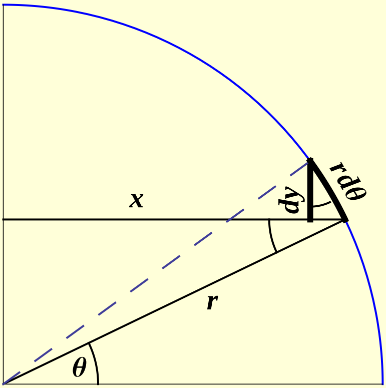
$$x^2 + y^2 = r^2 \Longrightarrow x\,dx + y\,dy = 0$$

$$ds^2 = r^2\,d\theta^2 = dx^2 + dy^2 = dx^2\left(1 + \frac{x^2}{y^2}\right) = r^2\,\frac{dx^2}{y^2},$$

$$d\theta^2 = \frac{dx^2}{y^2} = \frac{dy^2}{x^2} \qquad \Longrightarrow \qquad \begin{aligned} dy &= x\,d\theta \\ dx &= -y\,d\theta \end{aligned}$$

$$\Longrightarrow \qquad \begin{aligned} d\sin\theta &= \cos\theta\,d\theta \\ d\cos\theta &= -\sin\theta\,d\theta \end{aligned}$$

Proof Without Words



(with Aaron Wangberg)

$$dy = (r\,d\theta)\cos\theta$$
$$d(r\sin\theta) = r\,\cos\theta\,d\theta$$
$$d(\sin\theta) = \cos\theta\,d\theta$$

Tevian Dray,
College Math J. **44**, 17–23 (2013).

Infinitesimals

But so great is the average person's fear of the infinite that to this day calculus all over the world is being taught as a study of limit processes instead of what it really is: infinitesimal analysis. (Rudy Rucker)

... many mathematicians think in terms of infinitesimal quantities: apparently, however, real mathematicians would never allow themselves to write down such thinking, at least not in front of the children. (Bill McCallum)

Differentials

$$
\begin{aligned}
d(u + cv) &= du + c\,dv \\
d(uv) &= u\,dv + v\,du
\end{aligned}
$$

$$
\begin{aligned}
d\left(u^n\right) &= nu^{n-1}\,du \\
d\left(e^u\right) &= e^u\,du \\
d(\sin u) &= \cos u\,du \\
d(\cos u) &= -\sin u\,du \\
d(\ln u) &= \frac{1}{u}\,du
\end{aligned}
$$

Derivatives (& Integrals)

Derivatives:

$$\frac{d}{du}\sin u = \frac{d\sin u}{du} = \frac{\cos u\, du}{du} = \cos u$$

Chain rule:

$$\frac{d}{dx}\sin u = \frac{d\sin u}{dx} = \frac{\cos u\, du}{dx} = \cos u\,\frac{du}{dx}$$

Inverse functions:

$$q = \ln u \Longrightarrow e^q = u \Longrightarrow d(e^q) = e^q\,dq = du \Longrightarrow dq = \frac{du}{e^q} = \frac{du}{u}$$

Integrals:

$$\int 2x\,cos(x^2)\,dx = \int \cos u\,du = \int d(\sin u) = \sin u = \sin(x^2)$$

Coherence

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with d"

Tevian Dray & Corinne A. Manogue,
Putting Differentials Back into Calculus,
College Math. J. **41**, 90–100 (2010).

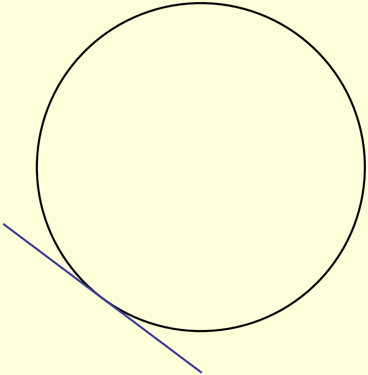
Example: Tangent line to a circle

Find the slope of the tangent line to the circle of radius 5 centered at the origin at the point $(-3,-4)$.

$$x^2 + y^2 = 25$$

$$\Longrightarrow 2x\,dx + 2y\,dy = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{x}{y} = -\frac{3}{4}$$



The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

- Small group activities
- Instructor's guide
- Online text (http://www.math.oregonstate.edu/BridgeBook)

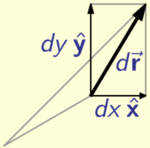
http://www.math.oregonstate.edu/bridge

DUE–0088901, DUE–0231032, DUE–0618877

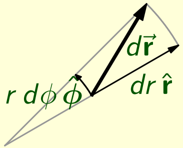
The Vector Calculus Bridge Project

Bridge Project homepage hits in 2009

Vector Differentials



$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

The Geometry of Gradient

$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$df \longleftrightarrow$ "small change in f"

$d\vec{\mathbf{r}} \longleftrightarrow$ "small step"

$$\begin{aligned} \text{level curve} \quad &\Longrightarrow \quad f = \text{constant} \\ &\Longrightarrow \quad df = 0 \\ &\Longrightarrow \quad \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0 \\ &\Longrightarrow \quad \vec{\nabla} f \perp \text{level curve} \end{aligned}$$

The gradient points "uphill"...

Raising Calculus to the Surface (Aaron Wangberg)

(Each surface is dry-erasable, as are the matching contour maps.)

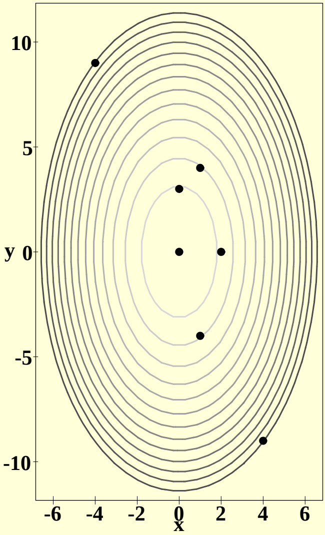
http://raisingcalculus.winona.edu

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



A Radical View of Calculus

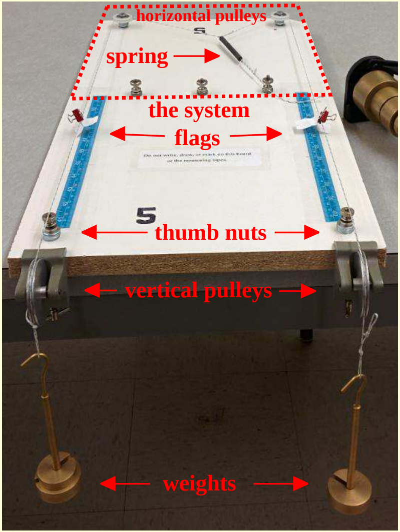
- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of physical quantities is not functions.

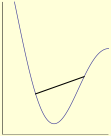
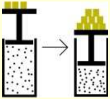
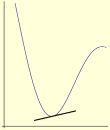
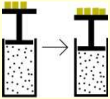
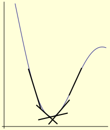
- Calculus is about infinitesimal reasoning.
- Derivatives are ratios of small quantities.
- The central idea of integrals is "chop, multiply, add".
- The central idea of curves and surfaces is "use what you know".
- The central representation of physical quantities is relationships (equations) between them.

Partial Derivatives Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.
- Mathematicians:
  "That's not a derivative!"

Paradigms in Physics Project
DUE–1023120, DUE–1323800



	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim\limits_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \ldots$	... depends on x	tedious repetition

An extended framework for the concept of the derivative.

No entry for symbolic differentiation!!

Captures roundoff error, measurement error, quantum mechanics...

"thick" derivatives

David Roundy et al., RUME Proceedings 2015, MAA, pp. 838–843.
(http://sigmaa.maa.org/rume/Site/Proceedings.html)

Tevian Dray, *Thick Derivatives*,
AMS Blog: On Teaching and Learning Mathematics (May 31, 2016).
(http://blogs.ams.org/matheducation/2016/05/31/thick-derivatives)

PUT DIFFERENTIALS BACK
into
CALCULUS!

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http://math.oregonstate.edu/bridge/papers