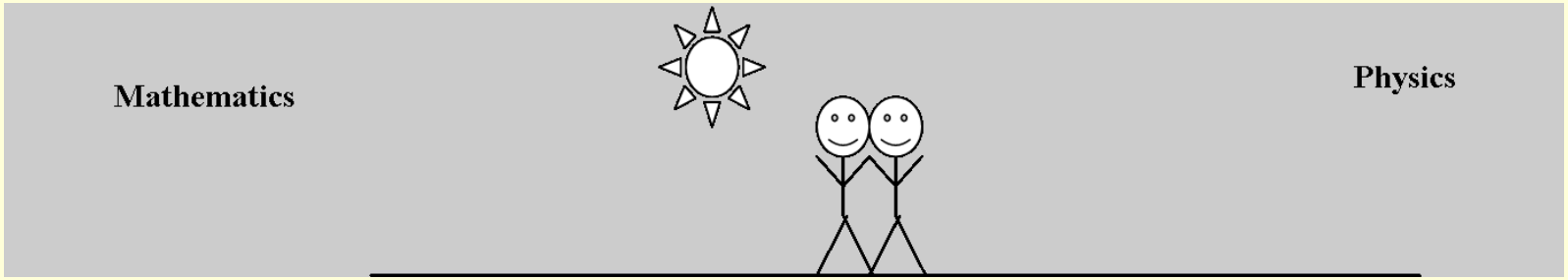
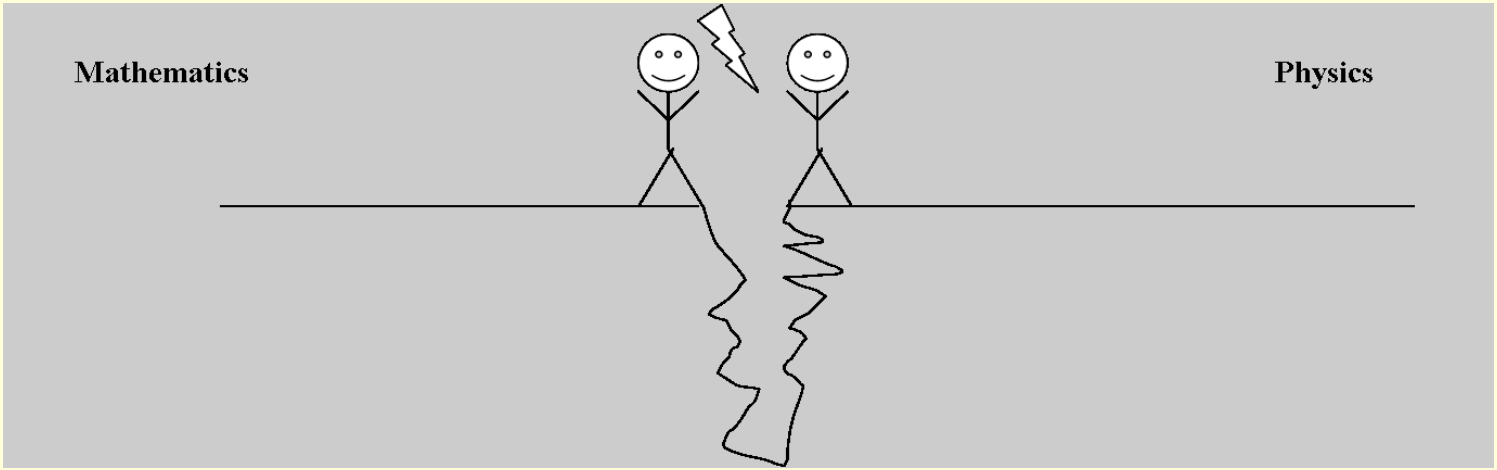
Bridging the Gap: Vector Calculus in Mathematics and Physics

Tevian Dray & Corinne A. Manogue

Mathematics vs. Physics

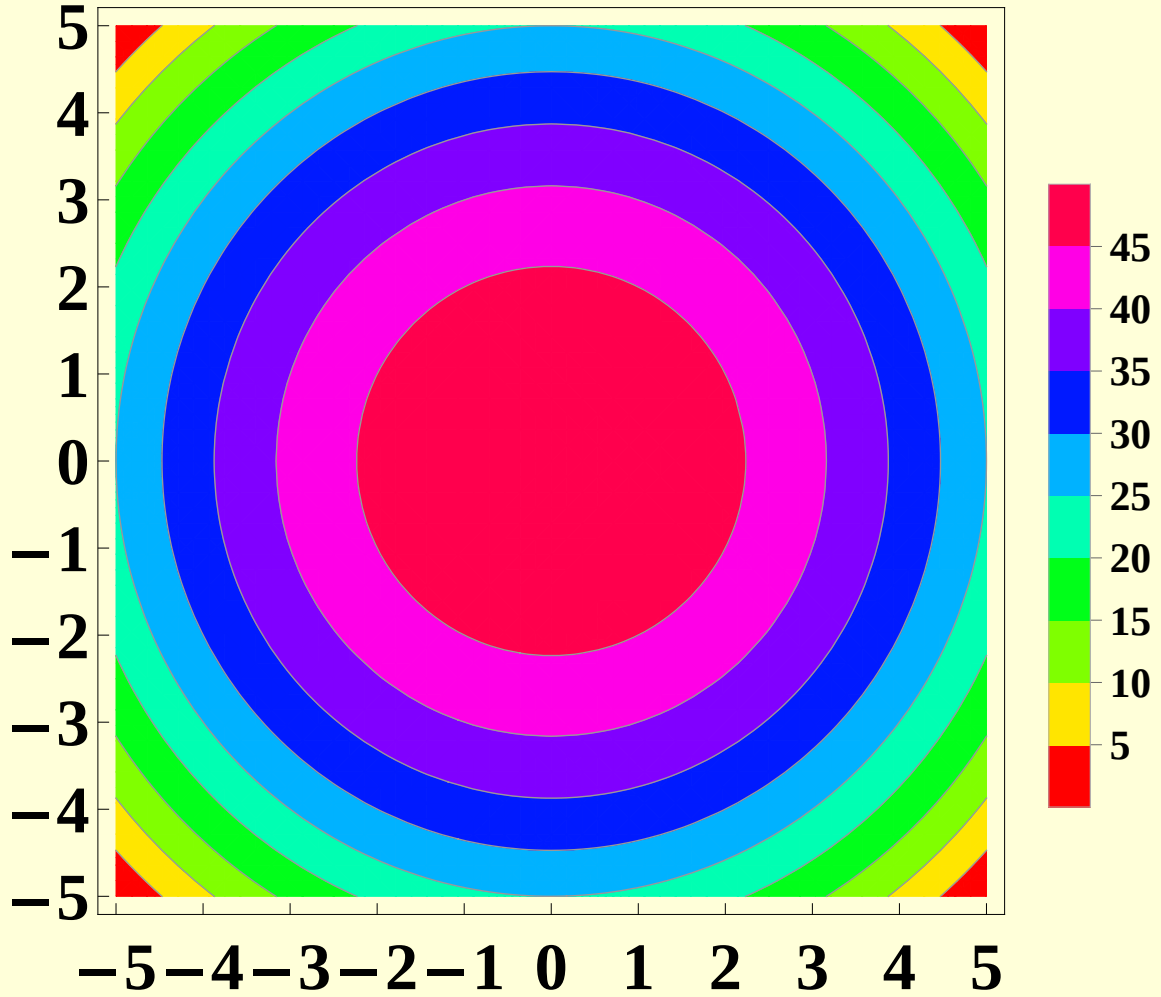




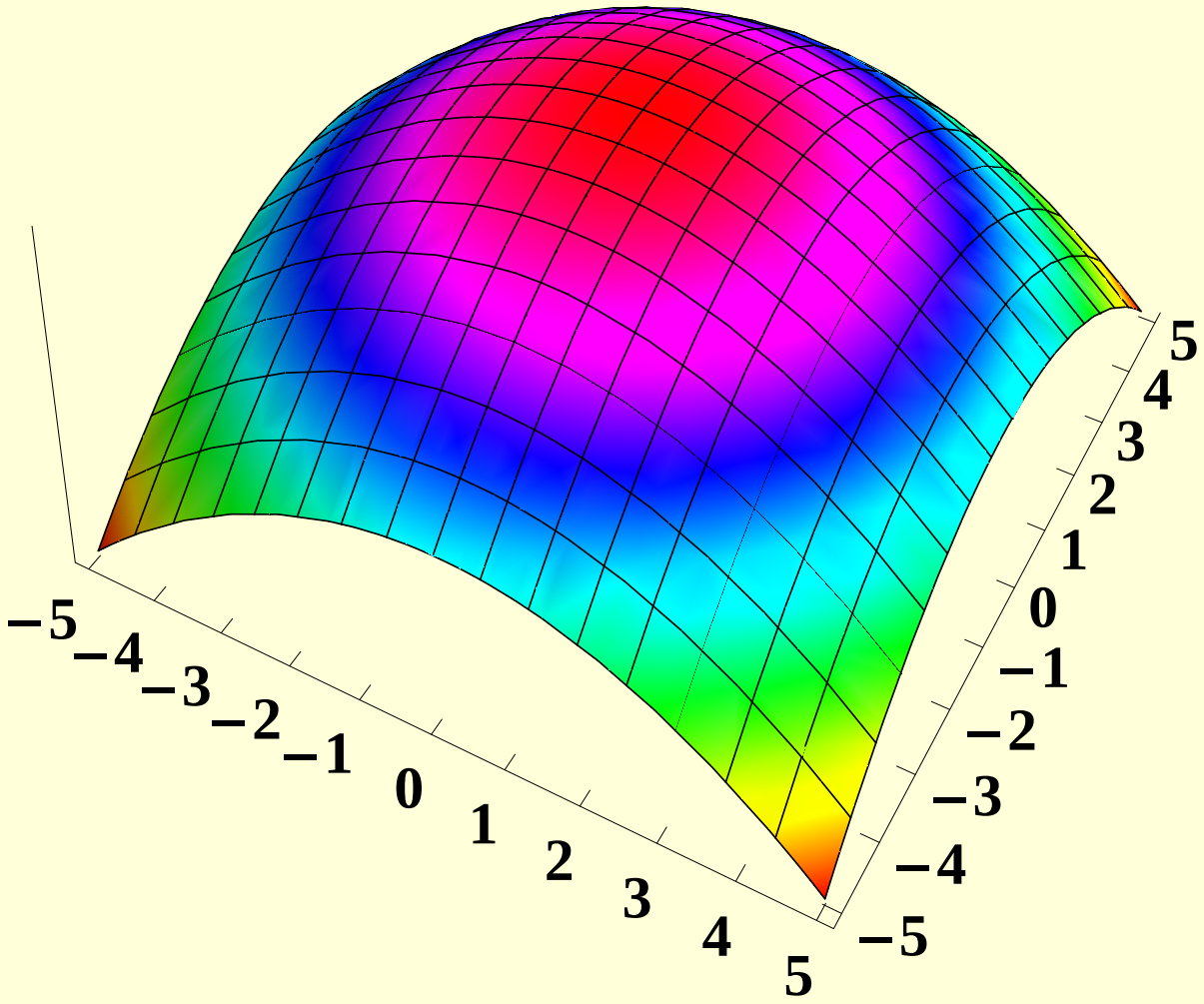
Table

0	9	16	21	24	25	24	21	16	9	0
9	18	25	30	33	34	33	30	25	18	9
16	25	32	37	40	41	40	37	32	25	16
21	30	37	42	45	46	45	42	37	30	21
24	33	40	45	48	49	48	45	40	33	24
25	34	41	46	49	50	49	46	41	34	25
24	33	40	45	48	49	48	45	40	33	24
21	30	37	42	45	46	45	42	37	30	21
16	25	32	37	40	41	40	37	32	25	16
9	18	25	30	33	34	33	30	25	18	9
0	9	16	21	24	25	24	21	16	9	0

Level Curves



Graph



What are Functions?

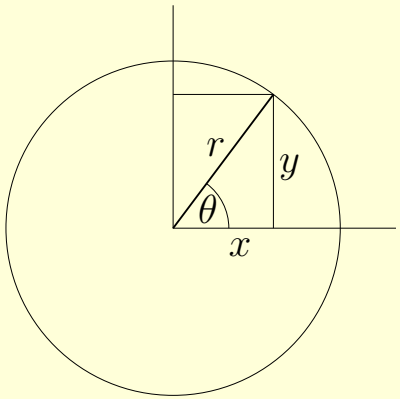
Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$



What are Functions?

MATH

$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

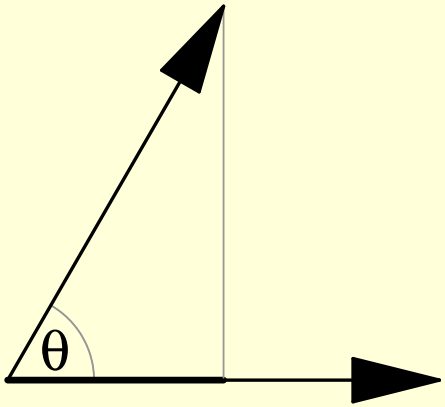
$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

Two disciplines separated by a common language...

Mathematics vs. Physics

- **Physics is about things.**
- **Physicists can't change the problem.**

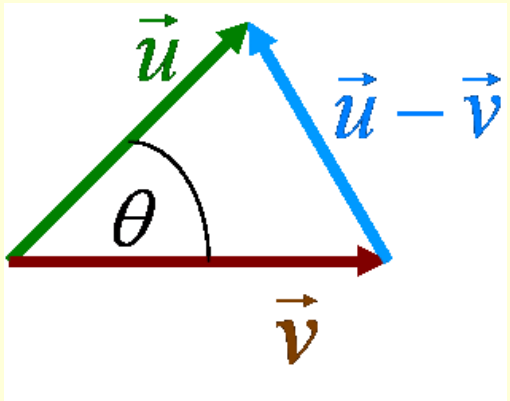
- **Mathematicians do algebra.**
- **Physicists do geometry.**



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$

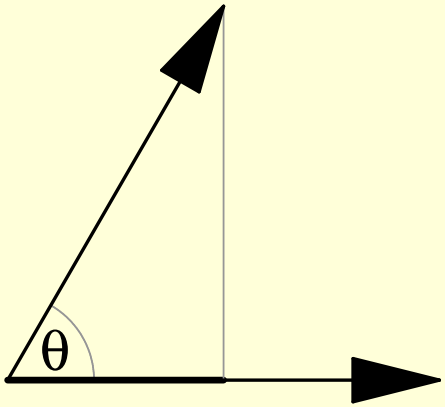


Law of Cosines:

$$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\text{“}c^2 = a^2 + b^2 - 2ab\cos\theta\text{”}$$



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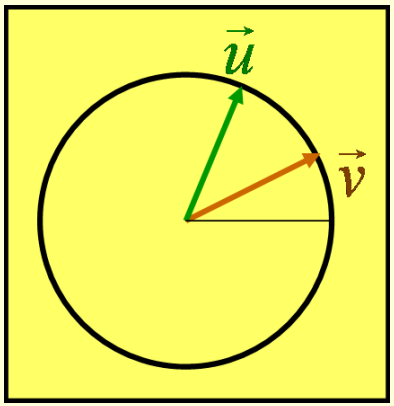
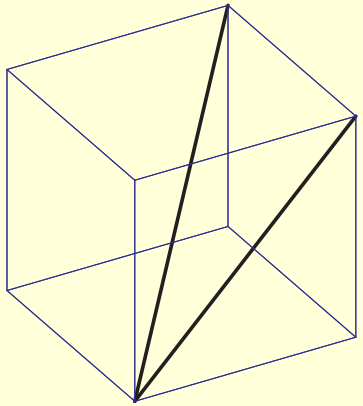


Addition Formulas:

$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\mathbf{x}} + \sin\beta\,\hat{\mathbf{y}}$$

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \cos(\alpha - \beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

Find the angle between the diagonal of a cube and the diagonal of one of its faces.



Algebra:

$$\begin{aligned}\vec{\mathbf{u}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\ \vec{\mathbf{v}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}}\end{aligned}$$

$$\implies \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2$$

Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta$$

$$\therefore \cos\theta = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{1}{\sqrt{6}}$$

Need both!

Teaching Geometric Reasoning

Vector Calculus Bridge Project:

`http://math.oregonstate.edu/bridge`

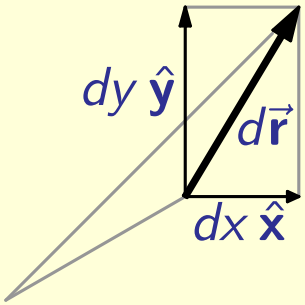
- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (`http://math.oregonstate.edu/BridgeBook`)

Paradigms in Physics Project:

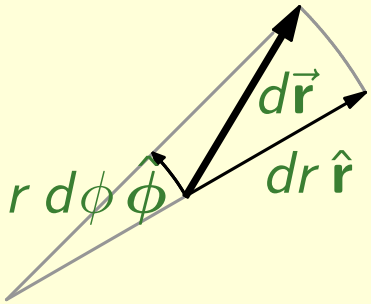
`http://physics.oregonstate.edu/portfolioswiki`

- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*

Infinitesimal Displacement



$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Gradient

Chain Rule:
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Differentials:
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
$$= \left(\frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}})$$

Master Formula:
$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$$f = \text{const} \implies df = 0 \implies \vec{\nabla} f \perp d\vec{\mathbf{r}}$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{d\vec{\mathbf{r}}}{|d\vec{\mathbf{r}}|}$$

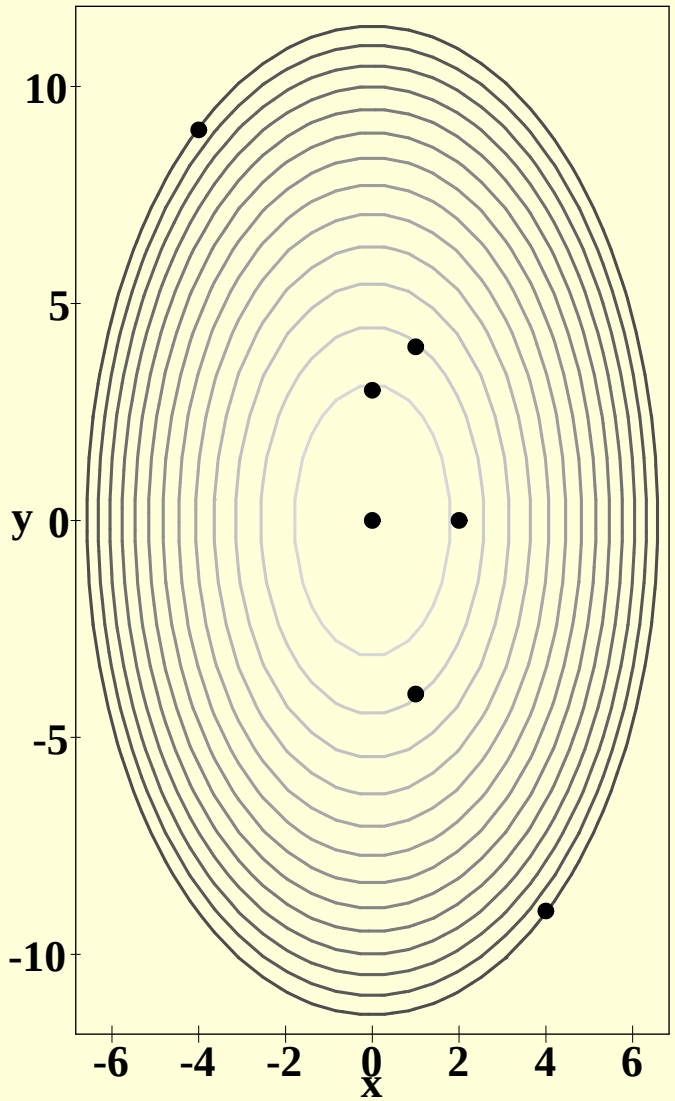
The gradient points in the steepest direction

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

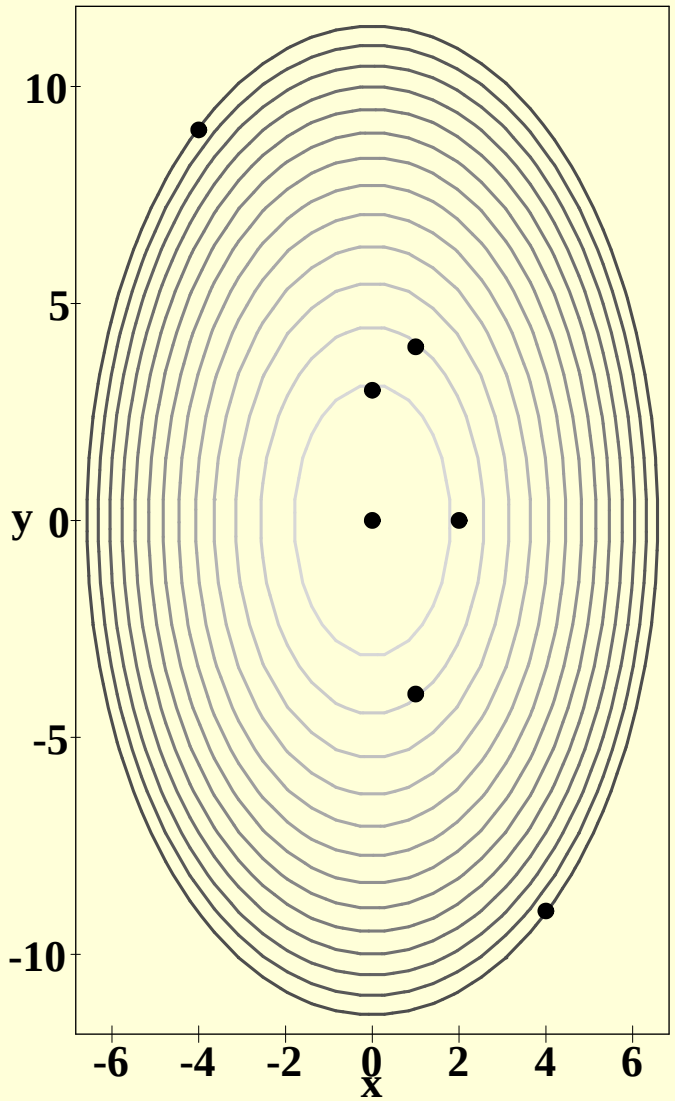
where $a = 5000$ ft, $b = 30 \frac{\text{ft}}{\text{mi}^2}$, and $c = 10 \frac{\text{ft}}{\text{mi}^2}$.



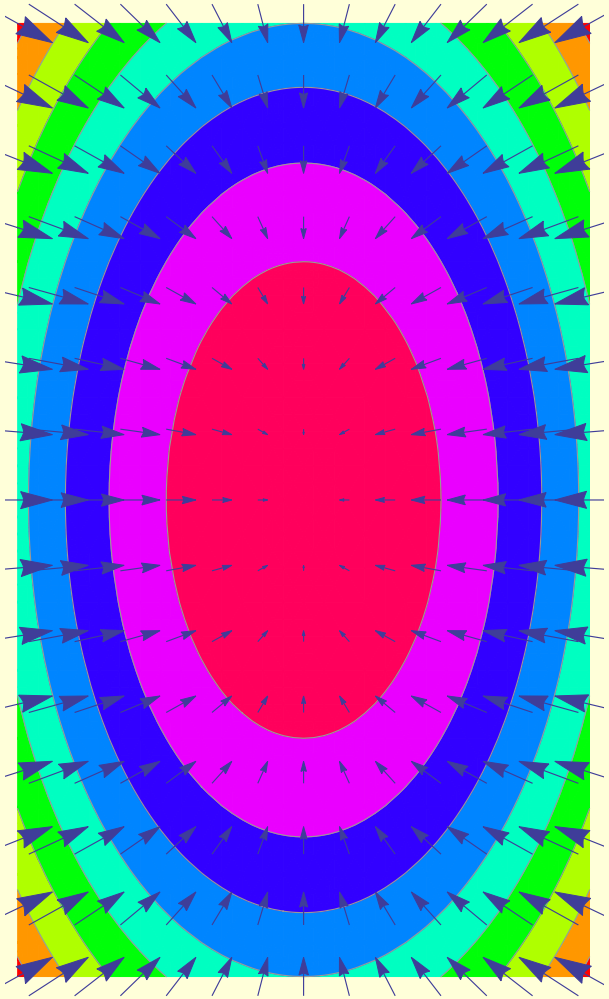
The Hill

Stand up and close your eyes.

Hold out your right arm in the direction of the gradient where you are standing.



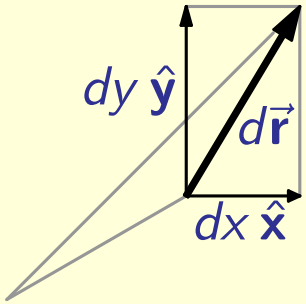
Visualization



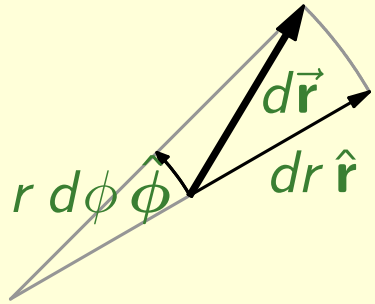
SUMMARY

Vector calculus is about one coherent concept:

Infinitesimal Displacement



$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

SUMMARY

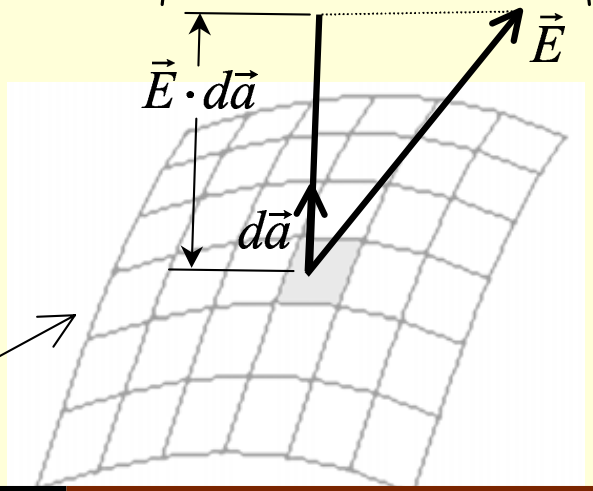
Flux is the total amount of electric field through a given area.

Perpendicular Component

Normal Vector

$$\Phi = \int \vec{E} \cdot d\vec{a}$$

$\sum$ over all rectangles



Kerry Browne (Ph.D. 2002)

SUMMARY

Geometry, geometry, geometry...

Tevian Dray & Corinne A. Manogue

http://math.oregonstate.edu/bridge
http://physics.oregonstate.edu/portfolioswiki