The Geometry of Relativity

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Differential Geometry

Definition

A *topological manifold* is a second countable Housdorff space that is locally homeomorphic to Euclidean space. A *differentiable manifold* is a topological manifold equipped with an equivalence class of atlases whose transition maps are differentiable.

General Relativity $\neq$ Differential Geometry

What math is needed for GR??

Background

- Differential geometry course: Rick Schoen
- GR reading course: MTW
- GR course: Sachs–Wu
- designed and taught undergrad math course in GR: Schutz, d'Inverno, Wald, Taylor–Wheeler, Hartle
- designed and taught undergrad physics course in SR
- NSF-funded curricular work (math and physics) since 1996
- national expert in teaching 2nd-year calculus

  http://blogs.ams.org/matheducation

geometer, relativist, curriculum developer, education researcher

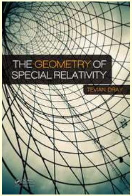
Mathematics, Physics, PER, RUME

Math vs. Physics

My math colleagues think I'm a physicsist.

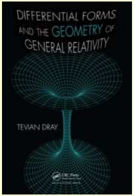
My physics colleagues know better.

Books



The Geometry of Special Relativity

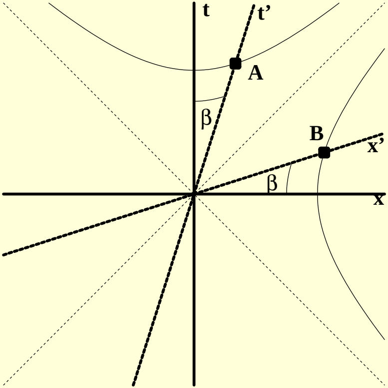
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
http://physics.oregonstate.edu/coursewikis/GSR

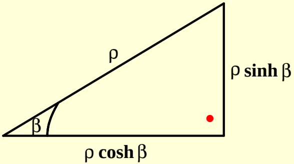


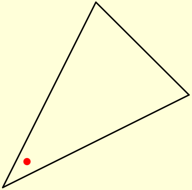
Differential Forms and the Geometry of General Relativity

Tevian Dray
A K Peters/CRC Press 2014
ISBN: 978-1-4665-1000-5
http://physics.oregonstate.edu/coursewikis/GDF
http://physics.oregonstate.edu/coursewikis/GGR

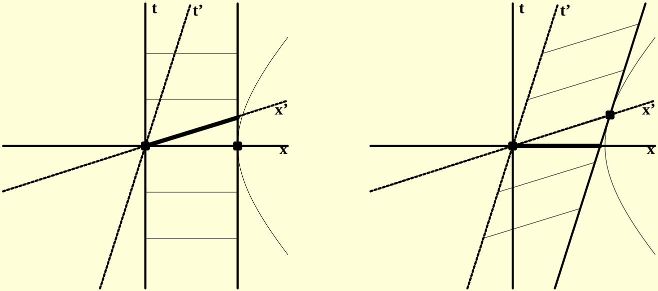
Trigonometry







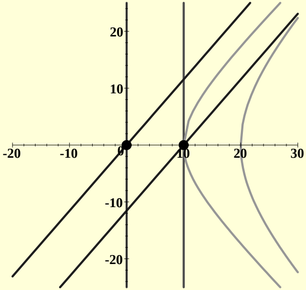
Length Contraction



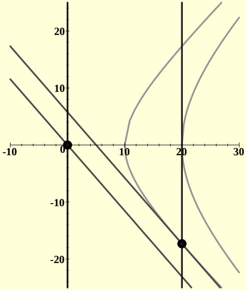
$$\ell' = \frac{\ell}{\cosh \beta}$$

Paradoxes

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn?



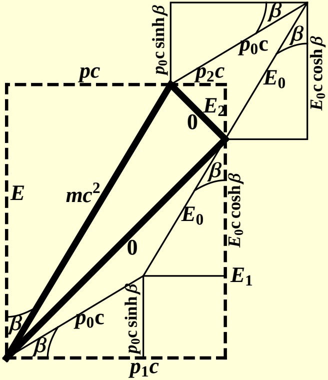
barn frame

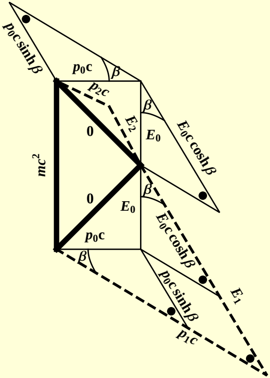


pole frame

Relativistic Mechanics

A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon. $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$





Addition Formulas

$$v = c \tanh \beta$$

Einstein Addition Formula:

$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta} \qquad \left(\text{“}v + w\text{”} = \frac{v+w}{1+vw/c^2}\right)$$

Conservation of Energy-Momentum:

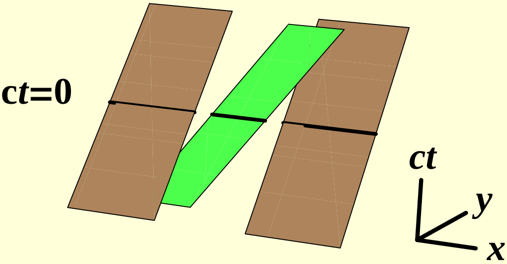
$$p = mc \sinh \alpha$$
$$E = mc^2 \cosh \alpha$$

Moving Capacitor:

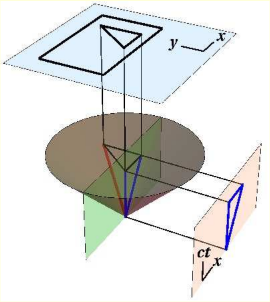
$$E'^y = C \cosh(\alpha + \beta) = E^y \cosh \beta - cB^z \sinh \beta$$
$$cB'^z = -C \sinh(\alpha + \beta) = cB^z \cosh \beta - E^y \sinh \beta$$

3d spacetime diagrams

(rising manhole)



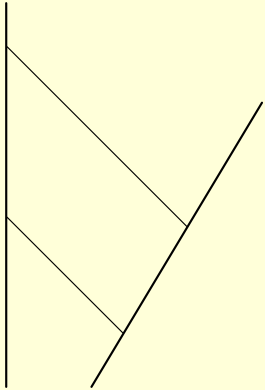
$(v\,\Delta t)^2 + (c\,\Delta t')^2 = (c\,\Delta t)^2$



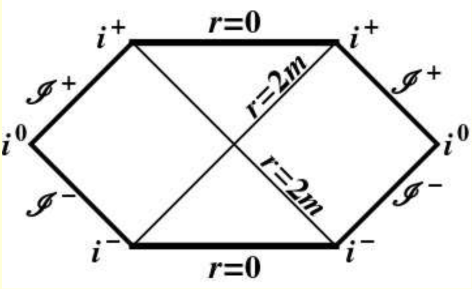
$(v\,\Delta t)^2 - (c\,\Delta t)^2 = -(c\,\Delta t')^2$

http://relativity.geometryof.org/GSR/book/updates/3d

The Geometry of General Relativity

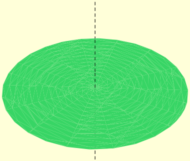


Doppler effect (SR)
Cosmological redshift (GR)

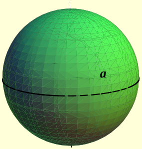


Asymptotic structure

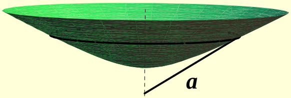
Line Elements



$$dr^2 + r^2\, d\phi^2$$



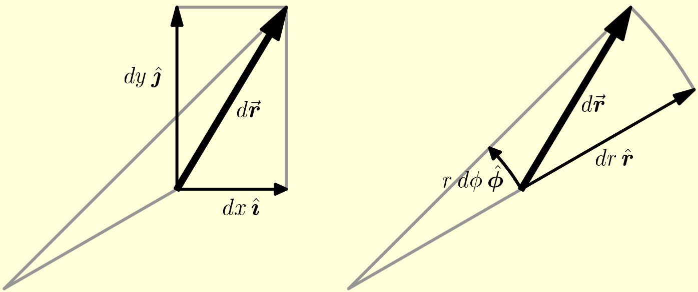
$$d\theta^2 + \sin^2\theta\, d\phi^2$$



$$d\beta^2 + \sinh^2\beta\, d\phi^2$$

Vector Calculus

$$ds^2 = d\vec{r} \cdot d\vec{r}$$



$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

Differential forms are integrands: $(*^2 = 1)$

$$f = f \qquad \text{(0-form)}$$

$$F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \qquad \text{(1-form)}$$

$$*F = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} \qquad \text{(2-form)}$$

$$*f = f\, dV \qquad \text{(3-form)}$$

Exterior derivative: $(d^2 = 0)$

$$df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}$$

$$dF = \vec{\nabla} \times \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}$$

$$d{*F} = \vec{\nabla} \cdot \vec{\boldsymbol{F}}\, dV$$

$$d{*f} = 0$$

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0$$
$$\vec{\nabla} \times \vec{B} - \dot{\vec{E}} = 4\pi\vec{J}$$

$$d{*}E = 4\pi\,{*}\rho$$
$$d{*}B = 0$$
$$dE + {*}\dot{B} = 0$$
$$dB - {*}\dot{E} = 4\pi\,{*}J$$

$$\vec{\nabla} \cdot \vec{J} + \dot{\rho} = 0$$

$$*d{*}J + \dot{\rho} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\Phi - \dot{\vec{A}}$$

$$B = {*}dA$$
$$E = -d\Phi - \dot{A}$$

Maxwell's Equations II

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} + \dot{\vec{B}} = 0$$
$$\vec{\nabla}\times\vec{B} - \dot{\vec{E}} = 4\pi\vec{J}$$

$$\vec{\nabla}\cdot\vec{J} + \dot{\rho} = 0$$

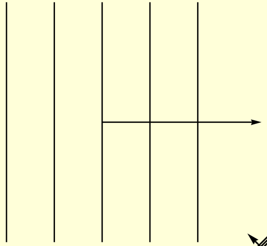
$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\Phi - \dot{\vec{A}}$$

$$F = \hat{E}\wedge dt + \hat{*}\hat{B}$$
$$*F = \hat{B}\wedge dt - \hat{*}\hat{E}$$
$$A = \hat{A} - \Phi\, dt$$

$$F = dA$$
$$d{*}F = 4\pi *J$$

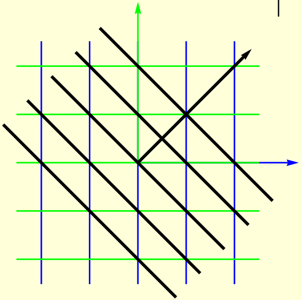
$$\Longrightarrow dF = 0$$
$$d{*}J = 0$$

The Geometry of Differential Forms

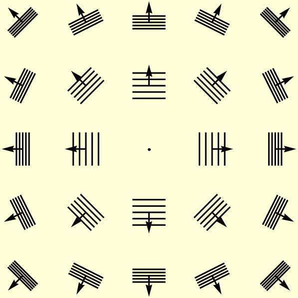


dx

$dx + dy$



$r\,dr = x\,dx + y\,dy$



Geodesic Equation

Orthonormal basis: $d\vec{r} = \sigma^i\, \hat{e}_i$ $\quad (\Longrightarrow ds^2 = d\vec{r} \cdot d\vec{r})$

Connection:

$$\omega_{ij} = \hat{e}_i \cdot d\hat{e}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{v}\, d\lambda = d\vec{r}$$

$$\dot{\vec{v}} = 0$$

Symmetry:

$$d\vec{X} \cdot d\vec{r} = 0$$

$$\Longrightarrow \vec{X} \cdot \vec{v} = \text{const}$$

Example: Polar Coordinates

Symmetry:

$$ds^2 = dr^2 + r^2\, d\phi^2$$
$$\Longrightarrow d\vec{\boldsymbol{r}} = dr\, \hat{\boldsymbol{r}} + r\, d\phi\, \hat{\boldsymbol{\phi}}$$
$$\Longrightarrow r\, \hat{\boldsymbol{\phi}} \text{ is Killing}$$

Idea: $df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} \Longrightarrow r\, \hat{\boldsymbol{\phi}} \cdot \vec{\nabla} f = \frac{\partial f}{\partial \phi} \Longrightarrow r\, \hat{\boldsymbol{\phi}} = \frac{\partial}{\partial \phi}$

Check: $d(r\, \hat{\boldsymbol{\phi}}) = dr\, \hat{\boldsymbol{\phi}} + r\, d\hat{\boldsymbol{\phi}} = dr\, \hat{\boldsymbol{\phi}} - r\, d\phi\, \hat{\boldsymbol{r}} \perp d\vec{\boldsymbol{r}}$

Geodesic Equation:

$$\vec{\boldsymbol{v}} = \dot{r}\, \hat{\boldsymbol{r}} + r\, \dot{\phi}\, \hat{\boldsymbol{\phi}} \Longrightarrow r\, \hat{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{v}} = r^2\, \dot{\phi} = \ell$$
$$\Longrightarrow 1 = \dot{r}^2 + r^2\, \dot{\phi}^2 = \dot{r}^2 + \frac{\ell^2}{r^2}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\boldsymbol{G}} = G^i\,\hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j\,\hat{\mathbf{e}}_i$$

$$\Longrightarrow d*\vec{\boldsymbol{G}} = 0$$

Field equation:

$$\boxed{\vec{\boldsymbol{G}} + \Lambda\,d\vec{\boldsymbol{r}} = 8\pi\,\vec{\boldsymbol{T}}}$$

(vector valued 1-forms, not tensors)

Stress-Energy Tensor

$$d\vec{r} = \sigma^a \, \hat{\mathbf{e}}_a$$

Vector-valued 1-form:

$$\vec{T} = T^a{}_b \, \sigma^b \, \hat{\mathbf{e}}_a$$

3-form:

$$\tau^a = *T^a$$

Conservation:

$$d\left(\tau^a \, \hat{\mathbf{e}}_a\right) = 0$$

$$*d*\vec{T} = 0$$

What about Tensors?

What tensors are needed to do GR?

Metric? Use $d\vec{r}$! (vector-valued 1-form!)
Curvature? Riemann tensor is really a 2-form. (Cartan!)
Ricci? Einstein? Stress-Energy? Vector-valued 1-forms!

$\exists$ only 1 essential symmetric tensor in GR!

Killing eq: $d\vec{X} \cdot d\vec{r} = 0$

Students understand line elements...

$$ds^2 = d\vec{r} \cdot d\vec{r}$$

Topic Order

Examples First!

- Schwarzschild geometry can be analyzed using vector calculus.
- Rain coordinates! (Painlevé-Gullstrand; freely falling)

Geodesics:

- EBH: Principle of Extremal Aging
- Hartle: variational principle (Lagrangian mechanics?)
- TD: "differential forms without differential forms"

Choices

Language:

- Mathematicians: invariant objects (no indices)
- Physicists: components (indices)
- Relativists: abstract index notation ("indices without indices")
- Cartan: curvature without tensors

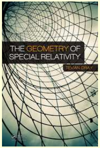
$\therefore$ use differential forms?

Coordinates:

- Mathematicians: coordinate basis (usually)
- Physicists: calculate in coordinates; interpret in orthonormal basis
- Equivalence problem: 79310 coordinate components reduce to 8690

$\therefore$ use orthonormal frames? ($d\vec{r}$?)

SUMMARY



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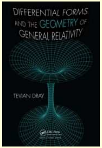

- Special relativity is hyperbolic trigonometry!
- General relativity can be described without tensors!
- BUT: Need vector-valued differential forms...

THE END