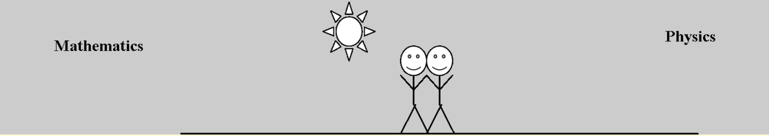
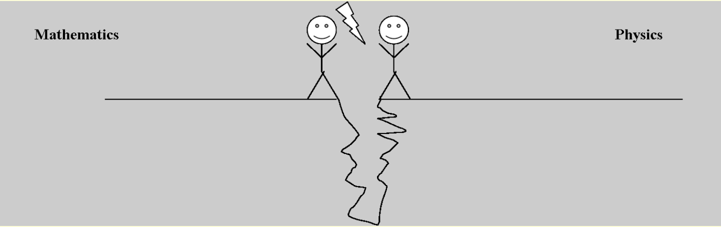
Using Geometric Reasoning to Teach Vector Calculus in Mathematics and Physics

Tevian Dray & Corinne Manogue

Oregon State University

Mathematics vs. Physics





Vector Calculus Bridge Project:

http://math.oregonstate.edu/bridge

- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (http://math.oregonstate.edu/BridgeBook)

Paradigms in Physics Project:

http://physics.oregonstate.edu/portfolioswiki

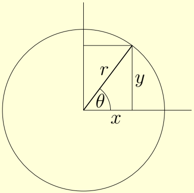
- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*

What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

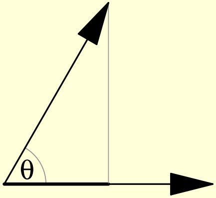
where k is a constant. What is $T(r, \theta)$?



A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$

Write something you know about the dot product on your small whiteboard.

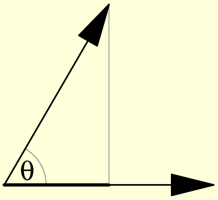


Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$

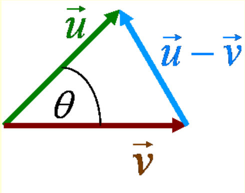
Dot Product



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



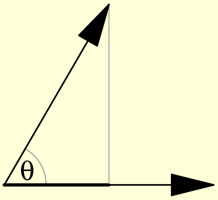
Law of Cosines:

$$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\text{“}c^2 = a^2 + b^2 - 2ab\cos\theta\text{”}$$

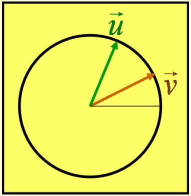
Dot Product



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



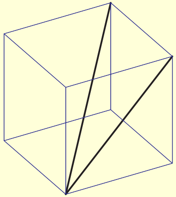
Addition Formulas:

$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\mathbf{x}} + \sin\beta\,\hat{\mathbf{y}}$$

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \cos(\alpha - \beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

Cube

Find the angle between the diagonal of a cube and the diagonal of one of its faces.



Algebra:

$$\begin{aligned}\vec{\mathbf{u}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\ \vec{\mathbf{v}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}}\end{aligned}$$

$$\Longrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2$$

Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta$$

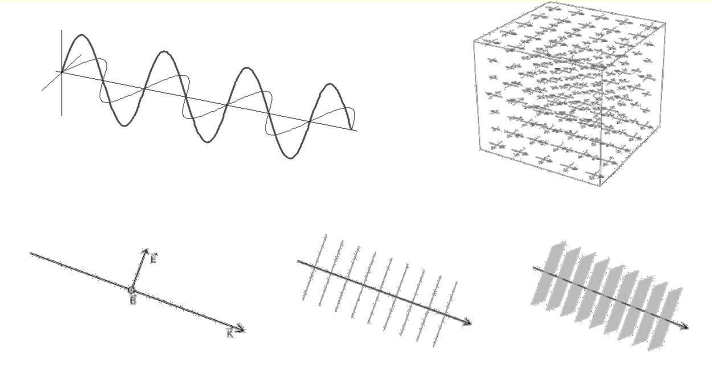
$$\therefore \cos\theta = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$$

Need both!

Compare and Contrast

- On your medium whiteboards, construct a square grid of points, approximately 2 inches apart, at least 7×7.
- I will draw an origin and a vector $\vec{\mathbf{k}}$ on your grid.
- For every point on your grid, imagine drawing the position vector $\vec{\mathbf{r}}$ to that point; calculate $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}$.
- Connect the points with equal values of $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}$.

Plane Wave Representations



Charge Density

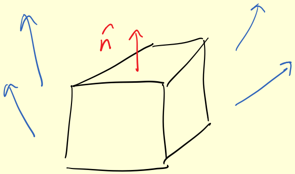
- Please stand up.
- Each of you is a point charge.
- Make a linear charge density.

Write something you know about flux on your small whiteboard.

Flux Demo

Divergence: Geometric Definition

$$\left.\frac{\left(\left(\vec{\mathbf{F}}\cdot\hat{\mathbf{n}}\right)dx\,dy\right)dz}{dz}\right|_{\text{top}} + \left.\frac{\left(\left(\vec{\mathbf{F}}\cdot\hat{\mathbf{n}}\right)dx\,dy\right)dz}{dz}\right|_{\text{bot}} = \frac{\partial F_z}{\partial z}d\tau$$



Flux per unit volume

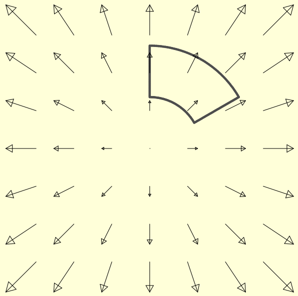
Schey, *div, grad, curl and all that*, Norton

Divergence: Small Group Activity

Coordinate independence of definition

$$\vec{\mathbf{F}} = r\,\hat{\mathbf{r}}$$

$$\vec{\nabla}\cdot\vec{\mathbf{F}} = \frac{1}{r}\frac{\partial}{\partial r}(r\,F_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

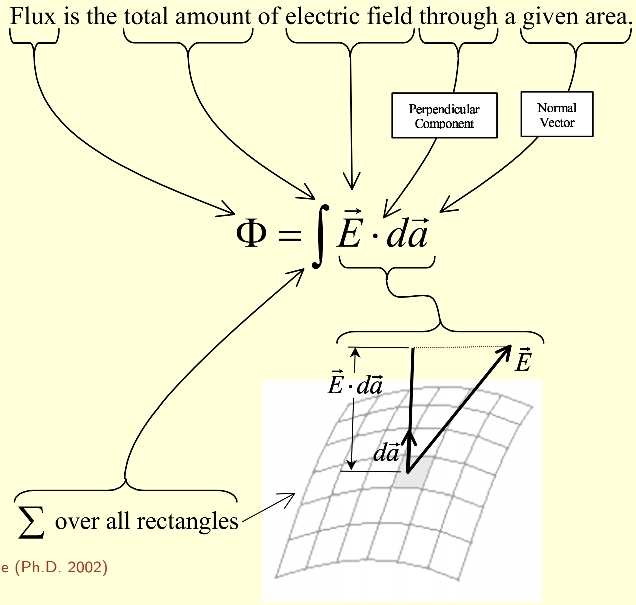


Add a physics law:

$$\oint_{\substack{\text{closed}\\\text{surface}}} \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

$$\Longrightarrow \vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

Multiple Representations



Kerry Browne (Ph.D. 2002)

Gradient

Write something you know about the gradient on your small whiteboard.

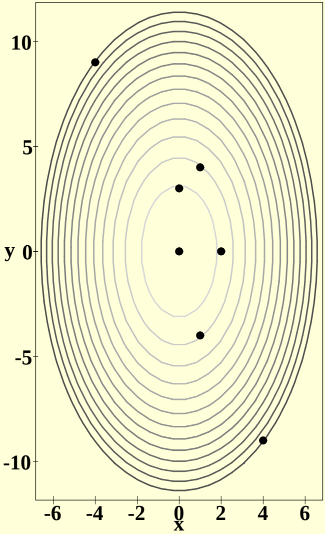
- $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + ...$
- The gradient points in the steepest direction.
- The magnitude of the gradient tells you how steep.
- The gradient is perpendicular to the level curves.

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

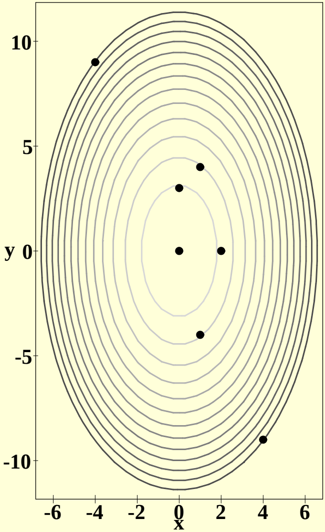
where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



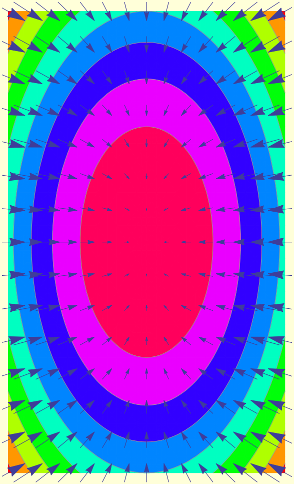
The Hill

Stand up and close your eyes.

Hold out your right arm in the direction of the gradient where you are standing.



Visualization



Infinitesimal Displacement

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$

$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$$

The Geometry of Gradient

Chain Rule: $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Differentials:
$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left(\frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}) \end{aligned}$$

Master Formula: $$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$$f = \text{const} \implies df = 0 \implies \vec{\nabla} f \perp d\vec{\mathbf{r}}$$

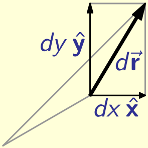
$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{d\vec{\mathbf{r}}}{|d\vec{\mathbf{r}}|}$$

The gradient points in the steepest direction

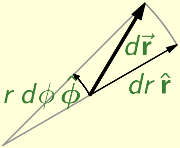
Vector Calculus

Vector calculus is about one coherent concept:

Infinitesimal Displacement



$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

Geometry, geometry, geometry...

Tevian Dray & Corinne A. Manogue

http://math.oregonstate.edu/bridge
http://physics.oregonstate.edu/portfolioswiki