Using Differentials in Thermodynamics

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Motivating Questions

Determine $\frac{\partial U}{\partial z}$ if

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x) + x^2$$

Determine $\frac{\partial h}{\partial g}$ if

$$g^3 + p^2 hf = 0$$
$$7pf + g = 0$$

Introduction

A typical question in thermodynamics:

Determine the *adiabatic bulk modulus* $\beta_S = -V \left(\frac{\partial p}{\partial V} \right)_S$

for an ideal gas with entropy

$$S = Nk \left(\ln \left[\frac{V}{N} \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \right] + \frac{5}{2} \right)$$

Ideal gas equation of state: $pV = NkT$

How to solve?

Notation

$$S = Nk\left(\ln\left[\frac{V}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{5}{2}\right)$$
$$pV = NkT$$

What are the variables??

$\hbar$, N, m, k are constants

S = entropy, p = pressure, V = volume, T = temperature

Any two of p, V, S, T are independent!

$$\left(\frac{\partial p}{\partial V}\right)_S$$

What does the S mean??

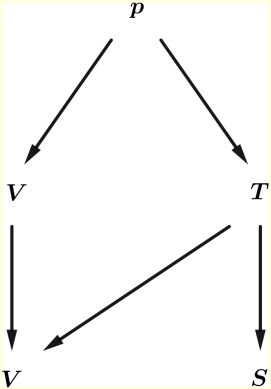
... with S held constant.

(always need to ask what's changing – and what isn't)

Chain Rule

$$pV = NkT$$
$$\implies p = p(V, T)$$

$$S = \ldots \frac{V}{N} \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \ldots$$
$$\implies T = T(V, S)$$



Nonlinear!

(Differentiate implicitly)

$$\left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S$$

Differentials

"Zap with d:" $xy = 1 \longmapsto x\,dy + y\,dx = 0$
Don't need to know which variables are independent!

Here (since $S =$ constant!):

$$0 = dS = \frac{3Nk}{2T}dT + \frac{Nk}{V}\,dV$$

Ideal gas law: $p\,dV + V\,dp = Nk\,dT$

Eliminating dT:

$$dp = -\frac{2kNT + 3pV}{3V^2}\,dV$$

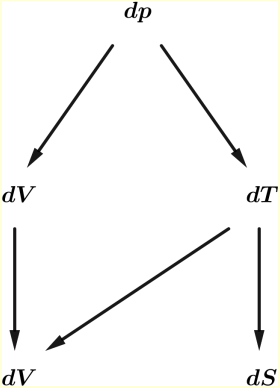
Thus: $\beta_S = \dfrac{5NkT}{3V}$

These equations are linear!

Chain Rule Diagrams

$$dp = \ldots\, dV + \ldots\, dT$$
$$dT = \ldots\, dV + \ldots\, dS$$

Linear!



Expert Reasoning

- Interviews with 10 experts across several disciplines.
- Task similar to above.
- No two experts used the same approach.

Three basic strategies ("epistemic games"):

- Substitution;
- Partial derivatives (multivariable chain rule);
- Differentials.

Kustusch, Roundy, Dray, Manogue (2012,2014);
Roundy, Weber, Dray, Bajracharya, Dorko, Smith, Manogue (2015).

Student Reasoning

- Interviews with ∼30 paradigms students.
- Similar quiz task, but purely symbolic.
- Similar task on final exam, in the context of thermodynamics.
- Analyzed using emergent coding scheme.

More finely grained classification of solution strategies:

- Variable susbstitution [Q:31%; F:11%];
- Differential substitution [Q:21%; F:44%];
- Implicit differentiation;
- Differential division;
- Chain rule diagrams [Q:21%; F:22%].
- Several other strategies that were unsuccessful.

All *except the first* had been discussed in class.

Followup study using thematic analysis:
Students are unfamiliar with Leibniz notation;
Students do not know how to eliminate extra dependent variables.

Founds, Emigh, Manogue (2017); Founds, Manogue (2022).

Multiple Representations

- Interviewed 8 students.
- Task: Determine partial derivative from a combination of graphical and numerical data.
- Analysis focused on students' ability to transform between representations.

Identified several classes of transformations:

- Translation;
- Consolodiation;
- Dissociation.

Introduced *Representational Transformation Diagram* (RTD)...

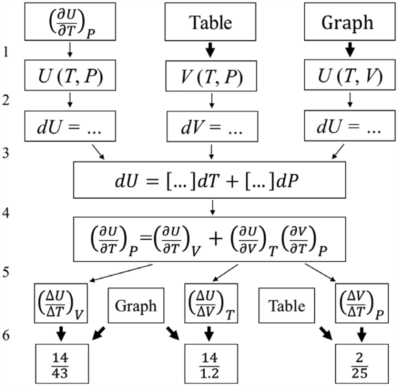
Bajracharya, Emigh, Manogue (2019)

Representational Transformation Diagram (RTD)

A flowchart to represent and analyze rich concept images.

- *Translation*
  (1 arrow in; 1 arrow out)
- *Consolidation*
  ($\geq$ 1 arrows in)
- *Dissociation*
  ($\geq$ 1 arrows out)

Length and complexity of RTD is proxy for cognitive load.



An RTD for the differentials method.

Bajracharya, Emigh, Manogue (2019)

SUMMARY

These results document the difficulties some students have based on their mathematical training when trying to master the expert reasoning around (partial) derivatives used in physics. They also document the existence of several expert approaches to the same task, both across disciplines and within a single discipline. Helping students become experts will require interdisciplinary coordination.