

Differential Forms

① Consider \mathbb{R}^n with coordinates $\{x^i\}$.

(can be any surface in \mathbb{R}^n)

② Consider the formal objects $\{dx^i\}$.

Idea: $df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial u^i} du^i$

\therefore treat $\{dx^i\}$ & $\{du^i\}$ as bases

& use calculus to change basis!

③ Let \mathcal{V} be the "almost vector space"

$$\mathcal{V} = \langle dx^i \rangle = \{a_i dx^i\}$$

Einstein summation convention!

"smooth" functions

What are the elements of \mathcal{V} ?

$$df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot d\vec{r}$$

"integrand of line integral corresponding to $\vec{\nabla} f$ "

Consistent: $d(x^\alpha) = \frac{\partial x^\alpha}{\partial x^i} dx^i$
 $= \delta_i^\alpha dx^i = dx^\alpha$

Kronecker delta

The elements of \mathcal{V} are 1-forms.
we write:

$$\mathcal{V} = \Lambda^1(\mathbb{R}^n) = \Lambda^1$$

Products of Differential Forms

$$\alpha, \beta \in \Lambda^1 \Rightarrow \alpha \wedge \beta \in \Lambda^2$$

Properties: $\beta \wedge \alpha = -\alpha \wedge \beta$

$(\alpha, \beta, \gamma \in \Lambda^1)$ $d \wedge d = 0$

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$$

$$f(\alpha \wedge \beta) = (f\alpha) \wedge \beta$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

Basis for 2-forms Λ^2 :

$$\{dx^i \wedge dx^j\} \quad 1 \leq i < j \leq n!$$

Basis for p-forms Λ^p : $0 \leq p \leq n$

$$\{dx^{i_1} \wedge \dots \wedge dx^{i_p}\} \quad 1 \leq i_1 < \dots < i_p \leq n$$

How many (independent) p-forms?

$$\dim \Lambda^p = \binom{n}{p} = \frac{n!}{p!(n-p)!} = \binom{n}{n-p}$$

$$\Rightarrow \dim \Lambda^0 = 1 = \dim \Lambda^n$$

$$\& \dim \Lambda^p = 0 \quad \text{for } p > n$$

Products of Forms, II

$$\alpha \in \Lambda^p, \beta \in \Lambda^q \Rightarrow \alpha \wedge \beta \in \Lambda^{p+q}$$

What is $\beta \wedge \alpha$?

Ex: $\alpha = dx$
 $\beta = dy \wedge dz$

⋮

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$$

why?