Divergence & Curl

$F \in \Lambda^1(\mathbb{R}^3)$:

$$d(F_x\,dx) = dF_x \wedge dx$$

$$= \left(\cancel{\frac{\partial F_x}{\partial x} dx} + \frac{\partial F_x}{\partial y} dy + \frac{\partial F_x}{\partial z} dz \right) \wedge dx$$

$$= \frac{\partial F_x}{\partial z} dz \wedge dx - \frac{\partial F_x}{\partial y} dx \wedge dy$$

$$\Longrightarrow dF = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx \wedge dy + \ldots$$

$$= \vec{\nabla} \times \vec{F} \cdot \hat{z}$$

$$= \vec{\nabla} \times \vec{F} \cdot d\vec{A}$$

$*F \in \Lambda^2(\mathbb{R}^3)$:

$$d(F_x\,dy \wedge dz) = dF_x \wedge dy \wedge dz$$

$$= \left(\frac{\partial F_x}{\partial x} dx + \ldots \right) \wedge dy \wedge dz$$

$$= \frac{\partial \,{}_x}{\partial x} dx \wedge dy \wedge dz$$

$$\Longrightarrow d{*F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz$$

$$= \vec{\nabla} \cdot \vec{F}$$

$$= \vec{\nabla} \cdot \vec{F}\; dx \wedge dy \wedge dz$$

Summary

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3$$

$$\Lambda^1 \overset{*}{\longleftrightarrow} \Lambda^2 \qquad \Lambda^0 \overset{*}{\longleftrightarrow} \Lambda^3$$

$\therefore$

$$\boxed{\begin{aligned} df &= \vec{\nabla} f \cdot d\vec{r} \\ F = \vec{F} \cdot d\vec{r} \Rightarrow \quad *dF &= \vec{\nabla} \times \vec{F} \cdot d\vec{r} \\ *d*F &= \vec{\nabla} \cdot \vec{F} \end{aligned}}$$

$\alpha \in \Lambda^1(\mathbb{R}^3) \Rightarrow$

$$\nabla \times \alpha = *d\alpha \quad \in \Lambda^1$$

$$\nabla \cdot \alpha = *d*\alpha \in \Lambda^0$$

The Laplacian

The Laplacian is

$$\boxed{\begin{aligned}\Delta f &= \vec{\nabla}\cdot\vec{\nabla} f \\ &= * d * d f\end{aligned}}$$

Ex: Rectangular coordinates

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$* df = \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx$$

$$\begin{aligned} d * df &= d\left(\frac{\partial f}{\partial x}\right)\wedge dy - d\left(\frac{\partial f}{\partial y}\right)\wedge dx \\ &= \frac{\partial^2 f}{\partial x^2}dx\wedge dy - \frac{\partial^2 f}{\partial y^2}dy\wedge dx \\ &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx\wedge dy \end{aligned}$$

$$\boxed{* d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}$$

Ex: Polar coordinates

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi \\ &= \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial \phi}\, r\,d\phi \end{aligned}$$

$$* df = \frac{\partial f}{\partial r}\, r\,d\phi - \frac{1}{r}\frac{\partial f}{\partial \phi}dr$$

$$\begin{aligned} d * df &= d\left(\frac{\partial f}{\partial r}r\right)\wedge d\phi - d\left(\frac{1}{r}\frac{\partial f}{\partial \phi}\right)\wedge dr \\ &= \frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)dr\wedge d\phi - \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial f}{\partial \phi}\right)d\phi\wedge dr \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}\right)dr\wedge r\,d\phi \end{aligned}$$

$$\boxed{* d * df = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}\right)}$$