HW #2

1. In this problem, you will determine the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of the Lie group $\mathrm{SL}(2,\mathbb{C})$ of invertible 2×2 matrices.

   (a) Suppose that $\gamma(\alpha)$ gives a (smooth) curve in $\mathrm{SL}(2,\mathbb{C})$ passing through the identity at $\alpha = 0$, with
   $$\gamma(\alpha) = \begin{pmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{pmatrix}.$$
   What are the values of $a(0)$, $b(0)$, $c(0)$, and $d(0)$?

   (b) Use the fact that $|\gamma(\alpha)| = \det(\gamma(\alpha)) = 1$ to determine a condition on the components, $a'(0)$, $b'(0)$, $c'(0)$, and $d'(0)$. What does this tell you about the form of a "typical" element $\gamma'(0)$ of $\mathfrak{sl}(2,\mathbb{C})$?

   (c) Your condition should tell you that $a'(0) + d'(0) = 0$, so $\mathfrak{sl}(2,\mathbb{C})$ is a subset of
   $$W = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$
   Verify that W is a vector space by checking that W contains the zero matrix and is closed under addition and scalar multiplication.

   (d) Find a basis for W, and show that each of your basis vectors is actually in $\mathfrak{sl}(2,\mathbb{C})$. *(To show a specific vector is in the Lie algebra of some Lie group G, you must find a (smooth) curve $\gamma(\alpha)$ in G such that $\gamma(0)$ is the identity and $\gamma'(0)$ equals the given vector.)*

   (e) Argue that $\mathfrak{sl}(2,\mathbb{C}) = W$.