	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

Table 1: The half-split 2×2 magic square of Lie algebras.

13 Counting the Magic Squares

Recall that the half-split 2×2 magic square of Lie algebras, shown in Table 1, consists entirely of real forms of orthogonal Lie algebras, that is, algebras of type $\mathfrak{a}$. In each case, the vector representation of the corresponding group can be obtained as the action

$$\boldsymbol{X} \longmapsto \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^{\dagger} \tag{276}$$

of unitary 2×2 matrices M over $\mathbb{K}' \otimes \mathbb{K}$ on matrices of the form

$$\boldsymbol{X} = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix} \tag{277}$$

with $A \in \mathbb{K}'$ and $a \in \mathbb{K}$. The determinant of $\boldsymbol{X}$ is the norm on the corresponding $\mathfrak{so}(p,q)$ and is preserved by such transformations, justifying the names $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})$ for these Lie algebras.

We can therefore count the dimension of these Lie algebras in two different ways. First of all, we have

$$p + q = \kappa + \kappa' \tag{278}$$

where $\kappa = |\mathbb{K}|$, $\kappa' = |\mathbb{K}'|$, so that

$$|\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K})| = |\mathfrak{so}(\kappa + \kappa')| = \frac{1}{2}(\kappa + \kappa')(\kappa + \kappa' - 1). \tag{279}$$

But we can also count the 2×2 anti-Hermitian matrices over $\mathbb{K}' \otimes \mathbb{K}$. We can put any of the $\kappa\kappa'$ elements of $\mathbb{K}' \otimes \mathbb{K}$ on the off diagonal, but the other off-diagonal element is then fixed. We can put any of the $(\kappa - 1) + (\kappa' - 1)$ *imaginary* elements of $\mathbb{K}' \otimes \mathbb{K}$ on the diagonal, but the usual determinant condition on special unitary matrices leads to the requirement that our matrices be tracefree, again determining the other, in this case diagonal, element. However, as we have seen, over $\mathbb{H}$ and $\mathbb{O}$ we must also consider some imaginary multiples of the identity matrix, generating the rotations ("phases")

$$\mathfrak{so}(\mathrm{Im}\mathbb{H}) \cong \mathfrak{so}(3), \tag{280}$$

$$\mathfrak{so}(\mathrm{Im}\mathbb{O}) \cong \mathfrak{so}(7), \tag{281}$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$1+0+0=1$	$2+1+0=3$	$4+3+3=10$	$8+7+21=36$
$\mathbb{C}'$	$2+1+0=3$	$4+2+0=6$	$8+4+3=15$	$16+8+21=45$
$\mathbb{H}'$	$4+3+3=10$	$8+4+3=15$	$16+6+6=28$	$32+10+24=66$
$\mathbb{O}'$	$8+7+21=36$	$16+8+21=45$	$32+10+24=66$	$64+14+42=120$

Table 2: Counting the half-split 2×2 magic.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{a}_{5(-7)}$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Table 3: The half-split 3×3 magic square of Lie algebras.

respectively. More generally, we set

$$s_\kappa = |\mathfrak{so}(\mathrm{Im}\mathbb{K})| = |\mathfrak{so}(\kappa-1)| = \frac{1}{2}(\kappa-1)(\kappa-2) = 0,0,3,21 \tag{282}$$

for $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, that is, for $\kappa = 1,2,4,8$. Putting this all together, we obtain

$$|\mathfrak{su}(2,\mathbb{K}'\otimes\mathbb{K})| = \kappa\kappa' + \big((\kappa-1)+(\kappa'-1)\big) + \big(s_\kappa + s_{\kappa'}\big) \tag{283}$$

which, after some algebra, agrees with (279). The resulting computations are shown explicitly in Table 2.

The half-split 3×3 magic square of Lie algebras, also known as the Tits–Freudenthal magic square, is shown in Table 3. The real forms are labeled both by their *rank* m (the dimension of the Cartan subalgebra), and by their *signature* $s = b-r$, the difference between the number of boosts and rotations. For example, the split real form $\mathfrak{f}_{4(4)}$ has a Cartan subalgebra of dimension 4, consisting entirely of boosts; the remaining elements divide evenly into rotations and boosts. Once one knows that $|\mathfrak{f}_4| = 52$, it is then easy to deduce that $\mathfrak{f}_{4(4)}$ contains 28 boosts and 24 rotations.

We can count the Lie algebras in the 3×3 magic square by counting anti-Hermitian matrices, as for the 2×2 magic square. There are now three independent off-diagonal elements, and two independent ways to put a given imaginary unit on the diagonal of a tracefree matrix. However, the number of "phases" is different.

Consider first the top row of the 3×3 magic square, whose entries are $\mathfrak{su}(3,\mathbb{K})$. By analogy with our treatment (276) of the 2×2 magic square, we consider the corresponding group action

$$\mathcal{X} \longmapsto \mathcal{M}\mathcal{X}\mathcal{M}^\dagger \tag{284}$$

of unitary 3×3 matrices $\mathcal{M}$ over $\mathbb{K}' \otimes \mathbb{K}$ on Hermitian matrices $\mathcal{X} \in \mathrm{H}_3(\mathbb{K})$; if $\mathbb{K} = \mathbb{O}$, $\mathcal{X}$ is an element of the Albert algebra. The unitary algebras $\mathfrak{su}(3, \mathbb{K})$ preserve the determinant of $\mathcal{X}$, so we need to ask which diagonal matrices with nonzero trace also do so.

Over $\mathbb{R}$, there are no diagonal, anti-Hermitian matrices. Over $\mathbb{C}$, the phase generated by i times the identity matrix commutes with all other matrices, and is therefore excluded in order to get a simple Lie algebra. In the 2×2 case over $\mathbb{H}$, however, we know that we must include the imaginary multiples of the identity matrix in order to get the $\mathfrak{so}(3)$ acting on $\mathrm{Im}\mathbb{H}$, and the tracefree, diagonal, imaginary matrices extend this $\mathfrak{so}(3)$ to the $\mathfrak{so}(4)$ acting on all of $\mathbb{H}$. In the 3×3 case over $\mathbb{H}$, we can also add complementary matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},$$

forming another copy of $\mathfrak{so}(3)$ that commutes with this $\mathfrak{so}(4)$. Thus, we must add 9 diagonal matrices in all, which we can rearrange into the 6 independent tracefree matrices, together with 3 phases. Over $\mathbb{O}$, however, this construction fails, since matrices of this complementary form fail to preserve the determinant when acting on the Albert algebra!

So where are the phases that we needed in the 2×2 case? They are still there, but can be generated by matrices of the form

$$\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix},$$

which are tracefree! We do of course still need to nest such matrices in order to obtain all of the rotations in $\mathrm{Im}\mathbb{O}$. However, not all 21 of the resulting nested combinations are independent, since the un-nested versions already generate 7 rotations, namely the ones rotating three planes at once. The 14 remaining nested elements are precisely the ones needed to generate $\mathfrak{g}_2$. Thus, over $\mathbb{H}$, we have 2 sets of 7 tracefree, diagonal, imaginary matrices, together with 14 additional, nested elements of $\mathfrak{g}_2$. [1]

Combining all of this information, we can generalize (283) to the 3×3 case, obtaining

$$|\mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})| = 3\kappa\kappa' + 2\big((\kappa - 1) + (\kappa' - 1)\big) + \big(d_\kappa + d_{\kappa'}\big) \tag{285}$$

where now

$$d_\kappa = 0, 0, 3, 14 \tag{286}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, that is, for $\kappa = 1, 2, 4, 8$. The resulting computations are shown explicitly in Table 4.

[1] We could equally well have generated nested elements using matrices of the form

$$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, strong triality implies that we get exactly the same elements of $\mathfrak{g}_2$ in either case.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$3+0+0=3$	$6+2+0=8$	$12+6+3=21$	$24+14+14=52$
$\mathbb{C}'$	$6+2+0=8$	$12+4+0=16$	$24+8+3=35$	$48+16+14=78$
$\mathbb{H}'$	$12+6+3=21$	$24+8+3=35$	$48+12+6=66$	$96+20+17=133$
$\mathbb{O}'$	$24+14+14=52$	$48+16+14=78$	$96+10+17=133$	$192+28+28=248$

Table 4: Counting the half-split 3×3 magic square.

There is another way to count the 3×3 magic square. Writing

$$\mathcal{X} = \begin{pmatrix} \boldsymbol{X} & \theta \\ \theta^\dagger & \phi \end{pmatrix} \tag{287}$$

and setting

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 1 \end{pmatrix} \tag{288}$$

demonstrates that $\mathfrak{su}(2, \mathbb{K}' \otimes \mathbb{K}) \subset \mathfrak{su}(3, \mathbb{K}' \otimes \mathbb{K})$. Furthermore, there are 3 natural ways to perform this embedding, depending on the location of 1 on the diagonal. Thus, we expect there to be 3 copies of the 2×2 algebra inside the corresponding 3×3 algebra.

There are two broad categories of matrices in these matrix algebras, namely diagonal matrices and "off-diagonal" matrices, that is, matrices with zeros on the diagonal. Under the action (284), off-diagonal matrices mix up the diagonal and off-diagonal elements of $\mathcal{X}$. There are indeed 3 independent copies of these transformations, corresponding to the "3" in (285). However, the number of independent diagonal matrices depends on whether one can act separately on $\boldsymbol{X}$ and θ in (287).

Diagonal matrices $\mathcal{M}$ that leave $\boldsymbol{X}$ alone must have a block structure similar to (288), but with $\boldsymbol{M}$ equal to the identity matrix. Over $\mathbb{R}$, the only possibilities are to put ± 1 in the lower-right element of $\mathcal{M}$; we rule out -1, since that matrix is not connected to the identity matrix. Over $\mathbb{C}$, we can put $e^{i\alpha}$ in the lower-right element of $\mathcal{M}$, so there is 1 way to act on θ while leaving $\boldsymbol{X}$ invariant. Similarly, over $\mathbb{H}$, we can put any unit element of $\mathbb{H}$ in the lower-right element of $\mathcal{M}$, so there is an SO(3) action on θ leaving $\boldsymbol{X}$ invariant. Finally, over $\mathbb{O}$, as we have already seen, such matrices do not preserve the determinant of $\mathcal{X}$; the only way to act on θ while leaving $\boldsymbol{X}$ invariant is again to multiply θ by -1, which doesn't count.

Over $\mathbb{R}$, there are no diagonal elements of $\mathfrak{su}(3, \mathbb{K})$ to begin with. Over $\mathbb{C}$, the single diagonal, tracefree, imaginary element in the 2×2 case generates an $\mathfrak{so}(2)$, but in the 3×3 case we have just argued that there are only 2 such elements, rather than 3. Thus, we must subtract 1 in this case. Over $\mathbb{H}$, rather than 3 copies of $\mathfrak{so}(4)$, we get $\mathfrak{so}(4)$ from the 2×2 case, together with an additional copy of $\mathfrak{so}(3)$ in the lower-right, resulting in a total of $6 + 3 = 9$ diagonal elements rather than $3 \times 6 = 18$. Thus, we must subtract 9 in this case. Finally, over $\mathbb{O}$, there are *no* additional elements, but only a single copy of $\mathfrak{so}(8)$. Thus, we must subtract the missing 2 copies, that is, we subtract $2 \times 28 = 56$ in this case.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$3\times 1-0=3$	$3\times 3-1=8$	$3\times 10-9=21$	$3\times 36-56=52$
$\mathbb{C}'$	$3\times 3-1=8$	$3\times 6-2=16$	$3\times 15-10=35$	$3\times 45-57=78$
$\mathbb{H}'$	$3\times 10-9=21$	$3\times 15-10=35$	$3\times 28-18=66$	$3\times 66-65=133$
$\mathbb{O}'$	$3\times 36-56=52$	$3\times 45-57=78$	$3\times 66-65=133$	$3\times 120-112=248$

Table 5: Using the 2×2 magic square to count the 3×3 magic square.

Combining all of this information, we can count the 3×3 magic square as

$$|\mathfrak{su}(3,\mathbb{K}'\otimes\mathbb{K})| = 3|\mathfrak{su}(2,\mathbb{K}'\otimes\mathbb{K})| - (t_\kappa + t_{\kappa'}) \tag{289}$$

where

$$t_\kappa = 0,1,9,56 \tag{290}$$

for $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, that is, for $\kappa=1,2,4,8$. The resulting computations are shown explicitly in Table 5; the result agrees with Table 4, as of course it must.