11 Properties of Roots

In this section, we fill in the missing details when deriving the properties of the roots of a simple Lie algebra $\mathfrak{g}$.

We assume that a Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$ of simultaneously diagonalizable elements has been chosen, and that $\mathfrak{g}$ has been decomposed into eigenspaces

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

so that, for any $H \in \mathfrak{h}$, we have

$$[H, X_\alpha] = \alpha(H) X_\alpha \tag{194}$$

for a finite set of roots $R \subset \mathfrak{h}^*$. Since $\mathfrak{g}$ is simple, the Killing form B on $\mathfrak{g}$ is nondegenerate. As argued in the previous section,

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 = B(\mathfrak{g}_\alpha, \mathfrak{h}) \tag{195}$$

for $\alpha + \beta \neq 0$ (and $\alpha \neq 0$, since $0 \notin R$). Recall further that the Jordan–Chevalley decomposition shows that any linear operator can be divided uniquely into the sum of a diagonalizable operator and a nilpotent operator. Thus, nonzero elements of $\mathfrak{h}$ can *not* be nilpotent.

We begin with a simple lemma about traces. Since

$$\mathrm{tr}\,(XY) = \mathrm{tr}\,(YX), \tag{196}$$

the trace is cyclic, so that, for instance,

$$\mathrm{tr}\,(XYZ) = \mathrm{tr}\,(ZXY). \tag{197}$$

Thus,

$$\mathrm{tr}\,([X,Y]Z) = \mathrm{tr}\,(XYZ - YXZ) = \mathrm{tr}\,(XYZ - XZY) = \mathrm{tr}\,(X[Y,Z]) \tag{198}$$

which implies that

$$B([X,Y]Z) = B(X[Y,Z]). \tag{199}$$

Preferred Cartan Elements

Since B is nondegenerate, for any $X_\alpha \in \mathfrak{g}_\alpha$, there must be some $Y_\alpha \in \mathfrak{g}$ such that

$$B(X_\alpha, Y_\alpha) \neq 0, \tag{200}$$

and (195) now implies that $Y_\alpha \in \mathfrak{g}_{-\alpha}$. Thus, if α is a root, so is $-\alpha$, and we have

$$[H, Y_\alpha] = -\alpha(H) Y_\alpha. \tag{201}$$

Furthermore, since $\alpha \neq 0$, there must be some $H \in \mathfrak{h}$ such that, using (199,

$$B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) = B(X_\alpha, \alpha(H)Y) = \alpha(H) B(X_\alpha, Y_\alpha) \neq 0 \tag{202}$$

so that

$$2H_\alpha = [X_\alpha, Y_\alpha] \neq 0. \tag{203}$$

We now claim that $\alpha(H_\alpha) \neq 0$. If not, then

$$[H_\alpha, X_\alpha] = 0 = [H_\alpha, Y_\alpha]. \tag{204}$$

Consider now *any* representation of $\mathfrak{h}$, and suppose that the image of H_α, which we will also call H_α, has an eigenspace

$$V = \{v : H_\alpha v = \lambda v\} \tag{205}$$

for some fixed eigenvalue λ. Since X_α and Y_α commute with H_α, they both take eigenvectors to eigenvectors, that is, they take V to itself. So V itself is a representation of the subalgebra generated by $\{H_\alpha, X_\alpha, Y_\alpha\}$. But this means that, as matrices acting on V, we must have

$$[X_\alpha, Y_\alpha] = 2H_\alpha = 2\lambda \tag{206}$$

since representations preserve commutators (by definition), and H_α is a multiple of the identity matrix when acting on V. Taking the trace of both sides immediately forces $\lambda = 0$. Thus, the only eigenvalue of H_α is 0, which means that H_α is nilpotent. But the only nilpotent element of $\mathfrak{h}$ is 0, and $H_\alpha \neq 0$. This contradiction establishes the claim.

We can now rescale X_α and Y_α if necessary to obtain

$$\alpha(H_\alpha) = 1.$$

Although this construction does not determine X_α and Y_α uniquely, H_α is uniquely determined. These special elements of $\mathfrak{h}$ will be referred to as *preferred Cartan elements.*

Root Angles

As discussed in the previous section, $\{H_\alpha, X_\alpha, Y_\alpha\}$ form a standard basis for $\mathfrak{sl}(2,\mathbb{R})$, the split real form of $\mathfrak{su}(2)$. Thus, *all* representations of this Lie subalgebra of $\mathfrak{g}$ have half-integer eigenvalues, and in particular, $\beta(H_\alpha) \in \frac{1}{2}\mathbb{Z}$ for all roots β. The restriction of $\mathfrak{g}$ to real linear combinations of H_α, X_α, Y_α for all $\alpha \in R$ is therefore a real subalgebra of $\mathfrak{g}$, and is in fact the split real form of $\mathfrak{g}$.

As before, choose $T_\alpha \in \mathfrak{h}$ to be the element determined by

$$\alpha(H) = B(T_\alpha, H) \tag{207}$$

for all $H \in \mathfrak{h}$. We need to verify that T_α is a multiple of H_α.

Using (202) and (203, we have

$$B(2H_\alpha, H) = \alpha(H) B(X_\alpha, Y_\alpha) \tag{208}$$

so that in particular

$$B(2H_\alpha, H_\alpha) = \alpha(H_\alpha) B(X_\alpha, Y_\alpha) = B(X_\alpha, Y_\alpha) \tag{209}$$

which is furthermore nonzero by assumption. Thus,

$$B(H_\alpha, H) = B(T_\alpha, H)B(H_\alpha, H_\alpha) = B\big(B(H_\alpha, H_\alpha)T_\alpha, H\big) \tag{210}$$

for all $H \in \mathfrak{h}$, and we conclude that

$$T_\alpha = \frac{H_\alpha}{B(H_\alpha, H_\alpha)}.$$

as claimed previously. It now follows immediately that

$$\alpha(H_\beta) = B(T_\alpha, H_\beta) = \frac{B(H_\alpha, H_\beta)}{B(H_\alpha, H_\alpha)} \in \frac{1}{2}\mathbb{Z}$$

leading to the angles and ratios discussed in the previous section.

Multiples of Roots, and Multiplicity

It remains to show that the only multiples of a root α that are roots are $\pm\alpha$, and that each root only occurs once, that is, that $|\mathfrak{g}_\alpha| = 1$.

We have shown that $\{H_\alpha, X_\alpha, Y_\alpha\}$ is a standard basis for $\mathfrak{sl}(2, \mathbb{R})$, so that X_α, Y_α act as raising and lowering operators, respectively, for H_α. Suppose that $Z \in \mathfrak{g}_\alpha$, so that

$$[H_\alpha, Z] = \alpha(H_\alpha)Z = Z. \tag{211}$$

We know that *all* representations of $\mathfrak{sl}(2, \mathbb{R})$ consist of integer or half-integer "ladders". So how does $\mathfrak{sl}(2, \mathbb{R})$ act on Z? Moving down the ladder,

$$Z_0 = [Y_\alpha, Z] \tag{212}$$

is an element of $\mathfrak{h}$, since Y_α decreases *all* the eigenvalues of Z by α, that is

$$[H, Z_0] = \big[H, [Y_\alpha, Z]\big] = \big[[H, Y_\alpha], Z\big] + = \big[Y_\alpha, [H, Z]\big] = (-\alpha + \alpha)[Y_\alpha, Z] = 0 \tag{213}$$

for any $H \in \mathfrak{h}$. Since B is positive-definite on $\mathfrak{h}$, we can expand Z_0 as

$$Z_0 = zH_\alpha + H_\perp \tag{214}$$

where

$$B(H_\alpha, H_\perp) = 0. \tag{215}$$

But

$$B(H_\alpha, H_\perp) = 0 \Longrightarrow B(T_\alpha, H_\perp) = 0 \Longrightarrow \alpha(H_\perp) = 0 \tag{216}$$

so that

$$\alpha(Z_0) = z\alpha(H_\alpha) + 0 = z. \tag{217}$$

Putting this all together, we have

$$[X_\alpha, Z_0] = -[Z_0, X_\alpha] = -\alpha(Z_0)X_\alpha = -zX_\alpha. \tag{218}$$

But moving back up the ladder has to yield a multiple of Z, and we conclude that Z is itself a multiple of X_α, thus confirming that $|\mathfrak{g}_\alpha| = 1$.

The argument against any other multiples of α other than $\pm\alpha$ being roots is similar. If $c\alpha$ is a root, then choosing $Z \in \mathfrak{g}_{c\alpha}$ leads to

$$[H_\alpha, Z] = (c\alpha)(H_\alpha)Z = cZ \tag{219}$$

which forces c to be half-integer. Suppose $c \in \mathbb{Z}$. Then we can move up or down the ladder from Z to some element $Z_1 \in \mathfrak{g}_\alpha$. By the argument above, Z_1 must be a multiple of X_α. But then the ladder collapses, and $c = \pm 1$. In particular, 2α can not be a root. But now $\frac{1}{2}\alpha$ also can not be a root, which rules out half-integer values of c.

12 Dynkin Diagrams

We have seen that any simple Lie algebra can be decomposed into a Cartan subalgebra $\mathfrak{h}$ and its eigenspaces $\mathfrak{g}_\alpha$. Furthermore, the commutators of elements in $\mathfrak{g}_{\pm\alpha}$ determine a collection of preferred elements $H_\alpha \in \mathfrak{h}$, and the angles between the H_α are tightly constrained. We show in this section that these constraints in turn impose additional constraints on the possible types of simple Lie algebras.

The roots $\alpha \in \mathfrak{h}^*$ share the same angles as the corresponding H_α. We first reduce the number of roots to an independent set, $\{\alpha_i\}$. Any other root can, of course be expressed as a linear combination of our chosen basis. The symmetry of the roots allows us to choose the basis so that the nonzero coefficients are always either strictly positive or strictly negative; the two types of roots are then called *positive* and *negative* roots, respectively, and the basis elements are called *simple* roots. For each root α, one of $\pm\alpha$ is positive. The positive roots have the property that they are all sums, not differences, of the simple roots; in fact, the coefficients turn out to be positive integers. It is not hard to see that the positivity property means that the angle between any two simple roots must be obtuse, rather than acute.

Recall that all pairs of roots satisfy

$$\frac{\alpha \cdot \beta}{\alpha \cdot \alpha} \in \frac{1}{2}\mathbb{Z}$$

so that the angle θ between the roots must satisfy

$$4\cos^2\theta \in \{0, 1, 2, 3, 4\}.$$

Since the simple roots $\alpha_i \in \mathbb{R}^n$ are independent, they satisfy

$$0 > \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_j} \in \frac{1}{2}\mathbb{Z} \tag{220}$$

and the angles θ_{ij} between roots must satisfy

$$4\cos^2\theta_{ij} = 0, 1, 2, 3. \tag{221}$$

A *Coxeter graph* is a graph with one dot representing each simple root, and with each pair of dots connected by $4\cos^2\theta_{ij}$ lines. Equation (220) also constrains the relative magnitudes of (non-orthogonal)roots; a *Dynkin diagram* is a Coxeter graph with the addition of arrows on the double and triple connections pointing from the longer root to the shorter.

We now derive several restrictions on the possible Dynkin diagrams. To do so, the only assumptions we need are that the $\alpha_i \in \mathbb{R}^n$ satisfy the conditions above. Most of these conditions apply both to Coxeter graphs and Dynkin diagrams.

There are at most $k - 1$ connections between k simple roots.

For the purpose of this assertion, a *connection* refers to a pair of roots with at least one line connecting them; multiple lines between a given pair of roots still count as a single connection.

We let

$$\alpha = \sum_1^k \frac{\alpha_i}{|\alpha_i|} \neq 0. \tag{222}$$

where the inequality follows from the independence of the simple roots. The magnitude of α is given by

$$\begin{aligned} 0 < \alpha \cdot \alpha &= \sum_{i,j} \frac{\alpha_i \cdot \alpha_j}{|\alpha_i||\alpha_j|} \\ &= k + 2\sum_{i<j} \frac{\alpha_i \cdot \alpha_j}{|\alpha_i||\alpha_j|} \\ &= k + \sum_{i<j} 2\cos\theta_{ij}. \end{aligned} \tag{223}$$

But either $\alpha_i \cdot \alpha_j = 0$, in which case there are no connections, or

$$4\cos^2\theta_{ij} \in \{1, 2, 3\}, \tag{224}$$

in which case

$$2\cos\theta_{ij} \leq -1. \tag{225}$$

Thus,

$$0 < k - \#\text{ of connections} \tag{226}$$

which is what we were trying to show.

An immediate corollary is that *there are no closed loops in a Coxeter graph.*

There are at most 3 lines at each point.

Suppose that a simple root α is connected to k other simple roots α_i. Since there are no cycles, none of the α_i can be connected to each other. Thus,

$$i \neq j \Longrightarrow \alpha_i \cdot \alpha_j = 0. \tag{227}$$

Since the simple roots are independent, we can extend $\{\alpha_i\}$ to an orthogonal basis $\{\alpha_0, \alpha_i\}$ of the span $\langle \alpha, \alpha_i \rangle$. We can expand α in this basis, yielding

$$\alpha = \sum_0^k \frac{\alpha \cdot \alpha_i}{\alpha_i \cdot \alpha_i} \alpha_i. \tag{228}$$

Since α is independent of the α_i, we must have

$$\alpha \cdot \alpha_0 \neq 0 \tag{229}$$

so that

$$|\alpha|^2 = \sum_0^k \frac{(\alpha \cdot \alpha_i)^2}{\alpha_i \cdot \alpha_i} > \sum_1^k \frac{(\alpha \cdot \alpha_i)^2}{|\alpha_i|^2}. \tag{230}$$

Thus,

$$\# \text{ of lines} = 4\sum_1^k \cos^2\theta_i = 4\sum_1^k \frac{(\alpha\cdot\alpha_i)^2}{|\alpha|^2|\alpha_i|^2} < 4 \tag{231}$$

so that the number of lines is strictly less than 4, as claimed.

Simple chains of roots can be replaced by a single root.

Suppose that there is a chain of single lines connecting α_i to α_{i+1} for $1 \le i \le k-1$, in which case

$$\frac{\alpha_i\cdot\alpha_{i+1}}{\alpha_i\cdot\alpha_i} = -\frac{1}{2}, \tag{232}$$

$$\alpha_1\cdot\alpha_1 = ... = \alpha_k\cdot\alpha_k = Q^2. \tag{233}$$

We claim that the entire chain can be replaced by

$$\alpha = \sum_1^k \alpha_i \tag{234}$$

with the result still being a valid Coxeter graph.

We first compute

$$\begin{aligned}\alpha\cdot\alpha &= \sum_{i,j}\alpha_i\cdot\alpha_j \\ &= \sum_i \alpha_i\cdot\alpha_i + 2\sum_{i<j}\alpha_i\cdot\alpha_j \\ &= \sum_1^k \alpha_i\cdot\alpha_i + 2\sum_1^{k-1}\alpha_i\cdot\alpha_{i+1} \\ &= kQ^2 - (k-1)Q^2 = Q^2\end{aligned} \tag{235}$$

so that α has the same magnitude as each of the α_i. Furthermore, if β is any other root, β can be connected to at most one of the α_i, in which case

$$\beta\cdot\alpha = \beta\cdot\alpha_i; \tag{236}$$

if not, $\beta\cdot\alpha = 0$. In either case, all of the conditions on the original roots continue to hold if the k roots α_i are replaced by the single root α, as claimed.

Allowed Diagrams

We can use these three properties to rule out several Coxeter graphs. First of all, we consider only *connected* graphs, as only such graphs correspond to simple Lie algebras. The simplest graphs correspond to n roots, connected in a single chain by single lines. This diagram is of type A; since there are n roots, $\mathfrak{h}$ is n-dimensional. The corresponding Lie algebras are therefore called $\mathfrak{a}_n$. All of the roots have the same magnitude.

Consider instead a single chain, but with at least one double line. If there are more than one, then collapsing the roots in between must yield an allowable graph. But such a graph has four lines at a single point, which is not allowed. Thus, there is at most one double line, and hence exactly one. In this case, the roots have one of two magnitudes, depending on which side of the double connection they are on. Now suppose that there is a branch point in the graph. Again, there can not be more than one, since otherwise the chain between them could be collapsed, again resulting in a graph with four lines. Similarly, there can not be both a branch point and a double line. We will discuss these cases in further detail below; the corresponding Lie algebras are of types B–F.

Finally, if two roots are connected by three lines, then no other lines are possible. Thus, there is only one Coxeter graph with three lines. This diagram is of type G; since there are two roots, $\mathfrak{h}$ is 2-dimensional. The corresponding Lie algebra is therefore called $\mathfrak{g}_2$.

Diagrams with a double link.

Consider two simple chains, p roots α_m on one side of a double link, and q roots β_k on the other, with the double link connecting α_p and β_q. Since all but one of the links are single lines, we know that

$$\alpha_m \cdot \alpha_m = P^2, \qquad \beta_n \cdot \beta_n = Q^2 \tag{237}$$

and we can assume without loss of generality that there are p "short" roots and q "long roots, so that $Q^2 = 2P^2$. We also know that

$$\frac{\alpha_m \cdot \alpha_{m+1}}{\alpha_m \cdot \alpha_m} = -\frac{1}{2}, \tag{238}$$

with a similar relation for the roots β_k. Setting

$$\alpha = \sum_1^p m \frac{\alpha_m}{|\alpha_m|}, \qquad \beta = \sum_1^q k \frac{\beta_k}{|\beta_k|}, \tag{239}$$

and using similar techniques as in the previous calculations, we can compute

$$\begin{aligned}
\alpha \cdot \alpha &= \sum_1^p m^2 - \sum_1^{p-1} m(m+1) \\
&= p^2 + \sum_1^{p-1} \big(m^2 - m(m+1)\big) \\
&= p^2 - \sum_1^{p-1} m \\
&= p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}
\end{aligned} \tag{240}$$

and similarly

$$\beta \cdot \beta = \frac{q(q+1)}{2}. \tag{241}$$

We also know that

$$4\cos^2\theta = 4\frac{(\alpha_p \cdot \beta_q)^2}{P^2Q^2} = 2 \tag{242}$$

so that

$$(\alpha_p \cdot \beta_q)^2 = \left(\frac{p\alpha_p}{P} \cdot \frac{q\beta_q}{Q}\right)^2 = p^2q^2\frac{(\alpha_p \cdot \beta_q)^2}{P^2Q^2} = \frac{1}{2}p^2q^2. \tag{243}$$

But

$$(\alpha \cdot \beta)^2 < (\alpha \cdot \alpha)(\beta \cdot \beta) \tag{244}$$

or, in other words,

$$\frac{1}{2}p^2q^2 < \frac{1}{4}p(p+1)q(q+1) \tag{245}$$

$$\Longrightarrow 2pq < pq + p + q + 1 \tag{246}$$

$$\Longrightarrow pq < p + q + 1 \tag{247}$$

$$\Longrightarrow (p-1)(q-1) < 2. \tag{248}$$

There are three ways to satisfy (248). If $p = 1$, then there is only one short root; these Lie algebras are of type B, and denoted $\mathfrak{b}_{p+1}$. If $q = 1$, then there is only one long root; these Lie algebras are of type C, and denoted $\mathfrak{c}_{q+1}$. Finally, if $p = q = 2$, we get a single, exceptional case, of type F, and denoted $\mathfrak{f}_4$.

Diagrams with a branch point.

Consider now three simple chains meeting at a branch point ψ, with $p-1$ roots α_m in one chain, $q-1$ roots β_k in the second, and $r-1$ roots γ_ℓ in the third. Since all of the links are single lines, we know that

$$\alpha_m \cdot \alpha_m = \beta_k \cdot \beta_k = \gamma_\ell \cdot \gamma_\ell = \psi \cdot \psi = Q^2. \tag{249}$$

As above, set

$$\alpha = \sum_1^p m\frac{\alpha_m}{|\alpha_m|}, \qquad \beta = \sum_1^q k\frac{\beta_k}{|\beta_k|}, \qquad \gamma = \sum_1^r \ell\frac{\gamma_\ell}{|\gamma_\ell|}. \tag{250}$$

The magnitudes of α, β, and γ can be computed as in the preceding case, taking into account that the chains (excluding ψ) now contain $p-1$, $q-1$, and $r-1$ roots, respectively, yielding

$$\alpha \cdot \alpha = \frac{p(p-1)}{2}, \qquad \beta \cdot \beta = \frac{q(q-1)}{2}, \qquad \gamma \cdot \gamma = \frac{r(r-1)}{2}. \tag{251}$$

The roots α_m, β_k, γ_ℓ are mutually orthogonal; as before, we can complete this set of roots to an orthogonal basis by adding ψ_0, with $\psi \cdot \psi_0 \neq 0$. Expanding ψ in terms of this basis, we get

$$\psi = \sum_1^{p-1} \frac{\psi \cdot \alpha_m}{\alpha_m \cdot \alpha_m} \alpha_m + \sum_1^{q-1} \frac{\psi \cdot \beta_k}{\beta_k \cdot \beta_k} \beta_k + \sum_1^{r-1} \frac{\psi \cdot \gamma_\ell}{\gamma_\ell \cdot \gamma_\ell} \gamma_\ell + \frac{\psi \cdot \psi_0}{\psi_0 \cdot \psi_0} \psi_0 \tag{252}$$

and therefore

$$\psi \cdot \psi > \sum_1^{p-1} \frac{(\psi \cdot \alpha_m)^2}{Q^2} \alpha_m + \sum_1^{q-1} \frac{(\psi \cdot \beta_k)^2}{Q^2} \beta_k + \sum_1^{r-1} \frac{(\psi \cdot \gamma_\ell)^2}{Q^2} \gamma_\ell. \tag{253}$$

In other words, if θ_α is the angle between the vectors ψ and α, etc., then

$$\cos^2 \theta_\alpha + \cos^2 \theta_\beta + \cos^2 \theta_\gamma < 1. \tag{254}$$

But

$$(\alpha \cdot \psi)^2 = \frac{(p-1)^2 (\alpha_{p-1} \cdot \psi)}{Q^2} = (p-1)^2 (\psi \cdot \psi) \cos^2 \theta_{p-1} = \frac{1}{4} (p-1)^2 |\psi|^2. \tag{255}$$

Thus,

$$\frac{(\alpha \cdot \psi)^2}{|\alpha|^2 |\psi|^2} = \frac{(p-1)^2/4}{p(p-1)/2} = \frac{p-1}{2p} = \frac{1}{2}(1 - \frac{1}{p}) \tag{256}$$

and similarly for β and γ, which, using (254), yields

$$\frac{3}{2} - \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < 1 \tag{257}$$

or in other words

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{258}$$

There are several ways to satisfy (258). If $q = r = 2$, we get the diagrams of type D, and the Lie algebras denoted $\mathfrak{d}_{p+2}$. The only other distinct possibilities are $r = 2$, $q = 3$, and $p \in \{3, 4, 5\}$. These three exceptional cases belong to class E, and are denoted $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$.

Special Cases

The four infinite families correspond to known symmetry groups. In terms of their Lie algebras, we have

$$\mathfrak{a}_n = \mathfrak{su}(n+1) = \mathfrak{su}(n+1, \mathbb{C}), \tag{259}$$

$$\mathfrak{b}_n = \mathfrak{so}(2n+1) = \mathfrak{su}(n+1, \mathbb{R}), \tag{260}$$

$$\mathfrak{c}_n = \mathfrak{sp}(2n) = \mathfrak{su}(n, \mathbb{H}), \tag{261}$$

$$\mathfrak{d}_n = \mathfrak{so}(2n) = \mathfrak{su}(2n, \mathbb{R}). \tag{262}$$

There are similar correspondences for four of the five exceptional cases, namely

$$\mathfrak{f}_4 = \mathfrak{su}(3, \mathbb{O}), \tag{263}$$
$$\mathfrak{e}_6 = \mathfrak{sl}(3, \mathbb{O}) = \mathfrak{su}(3, \mathbb{C} \otimes \mathbb{O}), \tag{264}$$
$$\mathfrak{e}_7 = \mathfrak{sp}(6, \mathbb{O}) = \mathfrak{su}(3, \mathbb{H} \otimes \mathbb{O}), \tag{265}$$
$$\mathfrak{e}_8 = \mathfrak{su}(3, \mathbb{O} \otimes \mathbb{O}), \tag{266}$$

but the last case is somewhat different, namely

$$\mathfrak{g}_2 = \mathrm{Der}(\mathbb{O}), \tag{267}$$

which at the group level says that

$$G_2 = \mathrm{Aut}(\mathbb{O}), \tag{268}$$

that is, G_2 is the group of automorphisms of the octonions. We will discuss all of these correspondences further below.

The allowable Dynkin diagrams can easily be found online. Writing out the first few cases, there are several that overlap. For instance,

$$\mathfrak{su}(2) \cong \mathfrak{a}_1 \cong \mathfrak{b}_1 \cong \mathfrak{so}(3), \tag{269}$$
$$\mathfrak{so}(5) \cong \mathfrak{b}_2 \cong \mathfrak{c}_2 \cong \mathfrak{su}(2, \mathbb{H}). \tag{270}$$

Somewhat more unexpectedly, we have

$$\mathfrak{su}(4) \cong \mathfrak{a}_3 \cong \mathfrak{d}_3 \cong \mathfrak{so}(6), \tag{271}$$

as well as

$$\mathfrak{so}(4) \cong \mathfrak{d}_2 \cong \mathfrak{a}_1 \oplus \mathfrak{a}_1 \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \tag{272}$$

showing explicitly that $\mathfrak{d}_2$, whose Dynkin diagram consists of two disconnected points, is not simple. To avoid these repetitions, one normally assumes that $n > 1$ for $\mathfrak{b}_n$, that $n > 2$ for $\mathfrak{c}_n$, and that $n > 3$ for $\mathfrak{d}_n$.

Finally, the Lie algebras $\mathfrak{e}_n$ correspond to the the cases described above by the parameter values $(p, q, r) = (n - 3, 3, 2)$ with $n = 6, 7, 8$. We could in principle consider the cases $n = 4, 5$ as (also) being of type E. We would then have the duplications

$$\mathfrak{e}_5 \cong \mathfrak{d}_5 \cong \mathfrak{so}(10), \tag{273}$$
$$\mathfrak{e}_4 \cong \mathfrak{a}_4 \cong \mathfrak{su}(5), \tag{274}$$

so this normally not done. However, it is worth pointing out that the nested sequence of Lie algebras

$$\mathfrak{so}(10) \subset \mathfrak{su}(5) \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8 \tag{275}$$

has a long history in physics, with each of the corresponding Lie groups being considered as candidates for the symmetry group of a so-called Grand Unified Theory, combining three of the four fundamental forces of nature, namely electromagnetism and the strong and weak nuclear forces.