2 SO(3)

Representations

The rotation group in three Euclidean dimensions is known as SO(3). Let's try to apply the same reasoning in three dimensions that we did in two dimensions.

There is certainly a matrix representation of SO(3). If we ignore the z-direction entirely, we can surely embed the xy-rotations of SO(2) in SO(3). Thus, we expect matrices of the form

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

to be in SO(3). Similarly, we can rotate about the x- or y-axis, rather than the z-axis, yielding

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}. \tag{23}$$

But what does the general element of SO(3) look like?

It turns out that all rotations in three dimensions can be represented as a single rotation about an arbitrary axis. (This statement fails in higher dimensions. Why?) Thus, one description of SO(3) involves choosing an axis, then rotating about that axis. The choice of axis is equivalent to the choice of a point on the 2-sphere $\mathbb{S}^2$, which can be described by its colatitude θ and its longitude ϕ.

Rotate the sphere so that the north pole points in this direction. One possibility is to first rotate the sphere about the y-axis by an angle θ, thus bringing the north pole to colatitude θ (while keeping the longitude zero). Rotating the sphere about the z-axis by ϕ then brings the north pole to the longitude ϕ.

But how do we then rotate the sphere about this new axis? Easy; do that rotation first. In other words, *before* moving the north pole, rotate the sphere about the z-axis by the desired angle ψ, *then* move the north pole to the desired location.

Thus, the general element of SO(3) can be expressed in terms of the *Euler angles* (θ, ϕ, ψ) as

$$\begin{aligned} R(\theta,\phi,\psi) &= R_z(\phi)R_y(\theta)R_z(\psi) \qquad (24) \\ &= \begin{pmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi & -\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi & \sin\theta\cos\phi \\ \cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\theta\sin\phi \\ -\cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

What a mess!

Properties

It seems clear from the above discussion that SO(3) is a 3-dimensional manifold. But which one?

First of all, all three Euler angles are periodic. So we can think of SO(3) as a basepoint (the axis of rotation, determined by (θ, ϕ)), and a rotation about that axis (determined by ψ). So we might suspect that $\mathrm{SO}(3) \cong \mathbb{S}^2 \times \mathbb{S}^1$. As we will see later, that's not quite right; SO(2) does indeed have the *local* structure of $\mathbb{S}^2 \times \mathbb{S}^1$, but turns out to be a *fibre bundle* rather than a direct product. This fibre bundle is the well-known Hopf fibration of $\mathbb{S}^3$ over $\mathbb{S}^1$, so we now suspect that $\mathrm{SO}(3) \cong \mathbb{S}^3$. But that's still not quite right, since rotations about antipodal directions are equivalent. We must therefore identify antipodal points on $\mathbb{S}^3$, and we finally conclude that $\mathrm{SO}(3) \cong \mathbb{R}P^3$.

Again, what a mess!

Derivatives

Let's try again. First of all, since SO(3) preserves the length of a vector $v \in \mathbb{R}^3$, by the same argument used for SO(2) we must have $M^T M = 1$ for every $M \in \mathrm{SO}(3)$. Again, the "S" tells us that $|M| = 1$.

What if we look at derivatives of $M(\theta, \phi, \psi)$, rather than the group elements themselves? But which derivatives?

It is straightforward to compute

$$r_z = \dot{R}_z = R_z'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

$$r_x = \dot{R}_x = R_x'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{26}$$

$$r_y = \dot{R}_y = R_y'(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{27}$$

These matrices live in the tangent space to SO(3) at the identity, and are linearly independent. They must therefore span this 3-dimensional vector space.

Alternatively, we have

$$R_z(\alpha) = M(0, 0, \alpha), \tag{28}$$

$$R_x(\alpha) = M(\alpha, 0, 0), \tag{29}$$

$$R_y(\alpha) = M\left(\alpha, -\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{30}$$

which suggests that the r_m can be associated with the derivative operators ∂_θ, ∂_ϕ, and ∂_ψ—evaluated at the identity element.

Recall that for SO(2) there was a correspondence between $M'(\alpha)$ and the vector field ∂_ϕ. Since R_z is just a 3-dimensional version of M, we still expect this correspondence to hold. Furthermore, since $\{r_z, r_x, r_y\}$ are obtained from each other by cyclic permutations of the coordinates, the same relationship should hold for the corresponding vector fields. In ordinary spherical coordinates, we have

$$\partial_\phi = x\,\partial_y - y\,\partial_x, \tag{31}$$
$$-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi = y\,\partial_z - z\,\partial_y, \tag{32}$$
$$\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi = z\,\partial_x - x\,\partial_z. \tag{33}$$

Since these vector fields have been defined cyclically, they should correspond to $\{r_x, r_y, r_z\}$, respectively.

Wait a minute. These vector fields live on $\mathbb{S}^2$! But we really want vector fields that are tangent to SO(3), which is locally $\mathbb{S}^3$...

Although we won't go through the details, extending the vector fields above to $\mathbb{S}^3$ is straightforward. The ϕ- and ψ-directions turn out *not* to be orthogonal; using Gram-Schmidt orthogonalization corrects this deficiency by adding appropriate ∂_ψ terms. The resulting correspondence is:

$$R'_z(\alpha) \longleftrightarrow \partial_\phi, \tag{34}$$
$$R'_x(\alpha) \longleftrightarrow -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi + \csc\theta\cos\phi\,\partial_\psi, \tag{35}$$
$$R'_y(\alpha) \longleftrightarrow \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi + \csc\theta\sin\phi\,\partial_\psi. \tag{36}$$

However, it is difficult to evaluate these expressions at the identity due to the coordinate singularities there.

What do these several expressions have in common?

Commutators

The answer lies in the *commutation relations* between different elements. The *matrix commutator* is straightforward, and is defined by

$$[A, B] = AB - BA \tag{37}$$

for two $(n \times n)$ matrices A and B. Direct computation shows that

$$[r_x, r_y] = r_z, \qquad [r_y, r_z] = r_x, \qquad [r_z, r_x] = r_y, \tag{38}$$

which may look familiar to those who have studied quantum mechanics. [1]

There is also a *vector fields commutator*, defined for two vector fields X and Y by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{39}$$

and it can now be checked that the two sets of vector fields given above both share the commutation structure of the matrices r_m (up to an annoying but conventional sign). For instance,

$$[y\,\partial_z - z\,\partial_y, z\,\partial_x - x\,\partial_z]f = -(x\,\partial_y - y\,\partial_x)f, \tag{40}$$

[1] Be warned that our derivatives r_m are *antisymmetric*, whereas physicists normally work with the *Hermitian* matrices $-ir_m$.

since all other terms cancel by reversing the order of differentiation. Thus,

$$[y\,\partial_z - z\,\partial_y, z\,\partial_x - x\,\partial_z] = -(x\,\partial_y - y\,\partial_x). \tag{41}$$

The point is that these commutators are *constant*, that these commutators tell us something about the structure of the group, and that, best of all, we can determine the commutators using the matrices r_m without worrying about vector fields at all.

Which brings us to one final point: Even though the tangent spaces to $\mathbb{S}^2$ are, of course, only 2-dimensional, the three vector fields given above are nonetheless independent. How can this be? The space of vector fields is *not* a vector space, since the coefficients are functions, rather than constants. On a Lie group, however, the vectors at the identity naturally extend to vector fields everywhere, and so long as we take *constant* linear combinations of these vector fields, we still obtain a vector space. In this sense, the 3-dimensional vector field machinery on SO(3) can be successfully rewritten in terms of vector fields on SO(2), as we have done above, even though the latter is only 2-dimensional.