THE GEOMETRY OF THE EXCEPTIONAL LIE GROUPS

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Only wimps specialize in the general case;
real scientists pursue examples.

(attributed to Beresford Parlett by Sir Michael Berry)

1 SO(2)

Representations

The rotation group in two Euclidean dimensions is known as SO(2). How many representations of this group can you think of?

The first representation of SO(2) we consider is in terms of 2×2 matrices, of the form

$$M(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1}$$

The "O" in SO(2) stands for *orthogonal.* Orthogonal matrices satisfy

$$M^T M = 1 \tag{2}$$

where T denotes matrix transpose, and where we write simply 1 for the identity matrix (rather than I, for which we will have another use). Such matrices preserve the (squared) *magnitude*

$$|v|^2 = v^T v \tag{3}$$

of a vector $v \in \mathbb{R}^2$, since

$$(Mv)^T (Mv) = (v^T M^T)(Mv) = v^T v. \tag{4}$$

The "S" in SO(2) stands for *special*, and refers to the additional condition that

$$|M| = \det(M) = 1. \tag{5}$$

Orthogonal matrices M with $|M| = 1$ are *rotations*; if $|M| = -1$, the only other possibility, they are *reflections.*

Our second representation of SO(2) is in terms of the complex numbers, of the form

$$w(\phi) = e^{i\phi}. \tag{6}$$

Such complex numbers have norm 1, that is

$$|w(\phi)|^2 = w\,\overline{w} = 1 \tag{7}$$

and preserve the magnitude $|z|$ of any complex number $z \in \mathbb{C}$, since

$$|wz| = |w|\,|z| = |z|. \tag{8}$$

Sound familiar?

Our third representation of SO(2) is purely geometric. Rotations are rigid transformations of $\mathbb{R}^2$, obtained by, well, rotating the plane through a given angle ϕ. In other words, the rotations in SO(2) are in one-to-one correspondence with the angles in the (unit) circle, that is, with the circle itself. Thus, SO(2) can be thought of as the circle $\mathbb{S}^1$.

Take a moment to compare and contrast these various representations of SO(2). What are their properties?

Properties

The geometric representation makes clear that SO(2) is a *group*; the composition of two rotations is another rotation. In matrix language, we have

$$M(\alpha+\beta) = M(\alpha)M(\beta) \tag{9}$$

and similarly

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \tag{10}$$

for complex numbers. Setting $\alpha = 0$ corresponds to the identity element, and setting $\beta = -\alpha$ leads immediately to inverse elements.

Our two algebraic representations are clearly closely related. The identification of $M(\phi)$ with $e^{i\phi}$ suggests the further identification

$$x + iy \longleftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

This identification seems even more reasonable after writing

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x1 + y\Omega \tag{11}$$

where again 1 denotes the identity matrix, and

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

Notice that $\Omega^2 = -1$!

We note for future reference both that

$$\Omega = M'(0) \tag{13}$$

and that

$$M(\phi) = e^{\Omega\phi} \tag{14}$$

where matrix exponentiation is formally defined in terms of a power series (which always converges).

Returning to our geometric representation, since SO(2) can be thought of as $\mathbb{S}^1$, it is a smooth manifold, that is, it is a smooth 1-surface (i.e. a curve), on which one can introduce coordinates (e.g. ϕ). Thus, SO(2) is *both* a group *and* a manifold; it is our first example of a *Lie group*.

In the language of vector calculus, we can introduce a vector field that is tangent to $\mathbb{S}^1$. One possible choice would be the unit vector tangent to the circle, often written $\widehat{\boldsymbol{\phi}}$. In the language of differential geometry, however, vector fields are interpreted as directional derivative operators, so that

$$\vec{\boldsymbol{v}}(f) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} f. \tag{15}$$

Since

$$\widehat{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{\nabla}} f = \frac{1}{r}\frac{\partial f}{\partial \phi} \tag{16}$$

we choose instead the tangent vector

$$r\,\widehat{\boldsymbol{\phi}} = x\,\widehat{\boldsymbol{y}} - y\,\widehat{\boldsymbol{x}} \tag{17}$$

Equivalently, as differential operators we choose

$$\partial_\phi = x\,\partial_y - y\,\partial_x \tag{18}$$

where we have introduced the notation ∂_q for $\frac{\partial}{\partial q}$.

What do tangent vectors look like in the complex representation of SO(2)? Take the derivative! We have

$$\frac{dw}{d\phi} = iw = ie^{i\phi} \tag{19}$$

What does this result mean geometrically?

Evaluate this derivative first at the identity element, where $\phi = 0$. At the point $z = 1$, this derivative is i. But the i-direction is vertical; this direction is tangent to the circle at $z = 1$. A similar argument works at any point on the circle; iw always represents the direction rotated $\frac{\pi}{2}$ counterclockwise from w—precisely the direction tangent to the circle.

Finally, consider the matrix representation of SO(2). Again, take the derivative, yielding

$$A = M'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{20}$$

At other points on the circle, we have

$$M'(\alpha) = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} = M(\alpha)A. \tag{21}$$

This relationship between the derivative of a path in the group at any point and its derivative at the identity element is a hallmark of the study of Lie groups, and allows us to study such groups by studying their derivatives at the identity element, a much simpler process.