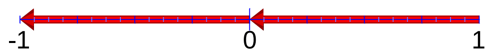


Figure 1: The root diagram of $\mathfrak{su}(2)$.

# 7 Representations of $\mathfrak{su}(2)$

We have the following basis elements for $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(2,\mathbb{C}') \cong \mathfrak{so}(2,1)$, a real form of $\mathfrak{su}(2)$:

$$\sigma_0 = \frac{1}{2}\sigma_z, \quad \sigma_\pm = \frac{1}{2}(\sigma_x \mp s_y) = \frac{1}{2}(\sigma_x \pm i\sigma_y), \tag{133}$$

with commutation relations

$$[\sigma_0, \sigma_\pm] = \pm\sigma_\pm, \quad [\sigma_+, \sigma_-] = 2\sigma_0. \tag{134}$$

These basis elements also form a basis of the *complexified* Lie algebra $\mathfrak{su}(2) \otimes \mathbb{C}$.

We can thus represent $\mathfrak{sl}(2,\mathbb{R})$ graphically as the points $0, \pm 1 \in \mathbb{R}$, representing $\sigma_z$ acting on itself and $\sigma_\pm$, respectively, connected by oriented arrows representing the action of $\sigma_\pm$, as shown in Figure 1. This diagram fully captures the algebraic description $\mathfrak{sl}(2,\mathbb{R})$ acting on itself, the so-called *adjoint representation* of $\mathfrak{sl}(2,\mathbb{R})$. Each of these statements can be reinterpreted as being about $\mathfrak{su}(2)\otimes\mathbb{C}$; Figure 1 is normally called the *root diagram of* $\mathfrak{su}(2)$.

We can now ask about more general representations of $\mathfrak{su}(2)$, with $\rho(\mathfrak{su}(2))$ acting on some vector space $V$. The commutation relations (134) show that $\sigma_0$ is diagonal in the given basis. It turns out that $L_z = \rho(\sigma_0)$ is diagonalizable in *any* representation $\rho$, [4] so we can choose a basis for $V$ consisting entirely of eigenvectors of $L_z$. If $w \neq 0$ is one such eigenvector, we have

$$L_z w = \lambda w \tag{135}$$

for some $w \in \mathbb{C}$. Letting $L_\pm = \rho(\sigma_\pm)$, we have

$$L_z L_\pm w = [L_z, L_\pm]w + L_\pm L_z w = \pm L_\pm w + L_\pm \lambda w = (\lambda \pm 1) L_\pm w \tag{136}$$

Thus, $L_\pm w$ is also an eigenvector of $L_z$, with eigenvalue $\lambda \pm 1$.

We want $V$ to be an *irreducible* representation of $\mathfrak{su}(2)$, by which we mean that there should be no (nonzero, proper) subrepresentations of $\mathfrak{su}(2)$ in $V$. Thus, acting repeatedly on $w$ with $L_\pm$ must generate a basis for $V$, as any vector not contained in the resulting span would itself generate a disjoint subrepresentation.

[4]This property holds for any *semisimple* Lie algebra, one for which the Killing form $B$ is nondegenerate, but is not true in general.

We also want $V$ to be *finite*. Since we are changing the eigenvalue at each step, this can only happen if there is a "biggest" eigenvalue. That is, we can assume without loss of generality that

$$L_+ w = 0 \tag{137}$$

and that the remaining basis vectors are obtained by repeated action of $L_-$.

We now compute

$$\begin{aligned} L_+ L_- w &= [L_+, L_-]w + L_- L_+ w = 2L_z w = 2\lambda w && (138)\\ L_+ L_- L_- w &= [L_+, L_-]L_- w + L_- L_+ L_- w \\ &= 2L_z L_- w + 2\lambda L_- w = 2(2\lambda - 1)L_w && (139)\\ \vdots \quad &= \quad \vdots \\ L_+ (L_-)^k w &= ... = \big(2k\lambda - k(k-1)\big)(L_-)^{k-1} w && (140) \end{aligned}$$

But for $V$ to be finite, $(L_-)^k w$ must be zero for some positive integer $k$. Assume that $k$ is the smallest such integer. Then $(L_-)^{k-1}$ is *not* zero, and therefore

$$2k\lambda - k(k-1) = 0 \tag{141}$$

by (140). Since $k \neq 0$, we conclude first of all that

$$\lambda = \frac{k-1}{2} \tag{142}$$

is an integer or half-integer, so that there are $k = 2\lambda + 1$ basis vectors, with eigenvalues

$$\lambda, \lambda - 1, ..., \lambda - 2\lambda = -\lambda. \tag{143}$$

We conclude that there is exactly one (irreducible) representation of $\mathfrak{su}(2)$ for each dimension $k \geq 2$, with eigenvalues $\{-\frac{k-1}{2}, ..., \frac{k-1}{2}\}$. Put differently, we can reproduce the commutation relations (134) using $n \times n$ matrices for any $n \geq 2$, and can do so in essentially just one way (up to change of basis).

# 8 $\mathfrak{su}(3)$

The unitary group SU(3) consists of all $3 \times 3$ unitary matrices with determinant 1, that is

$$\mathrm{SU}(3) = \{M \in \mathbb{C}^{3\times 3} : M^\dagger M = 1, |M| = 1\}. \tag{144}$$

The group SU(3) is the smallest of the unitary groups to be unrelated to the orthogonal groups; it's something new. As is the case for $\mathfrak{su}(2)$, the Lie algebra $\mathfrak{su}(3)$ consists of all $3 \times 3$ tracefree, anti-Hermitian matrices, that is

$$\mathfrak{su}(3) = \{A \in \mathbb{C}^{3\times 3} : A^\dagger + A = 0, \mathrm{tr}\,(A) = 0\}. \tag{145}$$

The standard basis for the *complexified* Lie algebra $\mathfrak{su}(3) \otimes \mathbb{C}$ consists of the *Gell-Mann matrices* [5]

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{146}$$

As with the Pauli matrices, the Gell-Mann matrices are Hermitian; unlike the Pauli matrices, they do not square to $\pm 1$. However, they are again orthonormal with respect to the Killing form (with overall normalization 2). We can obtain an anti-Hermitian basis of $\mathfrak{su}(3)$ itself (that is, not complexified) by using the matrices

$$\mu_m = -i\lambda_m \tag{147}$$

Alternatively, we can work with the real subset of these matrices, and study the real form $\mathfrak{sl}(3, \mathbb{R})$ of $\mathfrak{su}(3)$, that is

$$\mathfrak{sl}(3, \mathbb{R}) = \langle \lambda_1, \mu_2, \lambda_3, \lambda_4, \mu_5, \lambda_6, \mu_7, \lambda_8 \rangle \tag{148}$$

where $\langle ... \rangle$ denotes the *span* of the given elements, that is, the set of all linear combinations of these elements. By inspection, $\mathfrak{sl}(3, \mathbb{R})$ contains 5 boosts and 3 rotations.

[5] Our definition of $\lambda_5$ differs by an overall minus sign from the standard definition, in order to correct a minor but annoying lack of cyclic symmetry in the original definition.

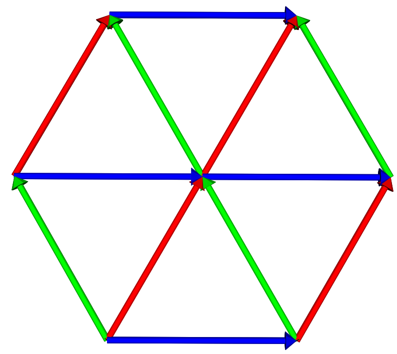

Figure 2: The root diagram of $\mathfrak{su}(3)$.

The given form of these matrices makes clear that $\lambda_3$ and $\lambda_8$ commute with each other, and that no larger set of basis elements will do so. The advantage of working with $\mathfrak{sl}(3,\mathbb{R})$ is that these real symmetric matrices have real eigenvalues; at the Lie algebra level, their real eigenvectors will lie inside the algebra, without the need for complexification.

Explicitly, we have the commutation relations

$$\begin{aligned}
&[\lambda_3, \lambda_8] = 0, \\
&\left[\lambda_3, \frac{1}{2}(\lambda_1 \mp \mu_2)\right] = \pm\frac{2}{2}(\lambda_1 \mp \mu_2), \quad \left[\lambda_8, \frac{1}{2}(\lambda_1 \mp \mu_2)\right] = 0, \\
&\left[\lambda_3, \frac{1}{2}(\lambda_4 \mp \mu_5)\right] = \mp\frac{1}{2}(\lambda_4 \mp \mu_5), \quad \left[\lambda_8, \frac{1}{2}(\lambda_4 \mp \mu_5)\right] = \mp\frac{\sqrt{3}}{2}(\lambda_4 \mp \mu_5), \\
&\left[\lambda_3, \frac{1}{2}(\lambda_6 \mp \mu_7)\right] = \mp\frac{1}{2}(\lambda_6 \mp \mu_7), \quad \left[\lambda_8, \frac{1}{2}(\lambda_6 \mp \mu_7)\right] = \pm\frac{\sqrt{3}}{2}(\lambda_6 \mp \mu_7).
\end{aligned} \tag{149}$$

Regarding the eigenvalues as vectors in $\mathbb{R}^2$, we can identify our basis elements with their eigenvalues, as follows:

$$\begin{aligned}
\frac{1}{2}(\lambda_1 \mp \mu_2) &\longleftrightarrow (\pm 2, 0), \\
\frac{1}{2}(\lambda_4 \mp \mu_6) &\longleftrightarrow (\mp 1, \mp 3), \\
\frac{1}{2}(\lambda_6 \mp \mu_7) &\longleftrightarrow (\mp 1, \pm 3),
\end{aligned} \tag{150}$$

and both $\lambda_3$ and $\lambda_8$ correspond to $(0,0)$. As with $\mathfrak{su}(2)$, we recover almost all of the structure of the Lie algebra by plotting these points. The result is shown in Figure 2, and is called the *root diagram of* $\mathfrak{su}(3)$. Each family of parallel lines represents the action of one of the three pairs of eigenvectors on the other eigenvectors; again, the eigenvectors can be thought of as raising and lowering operators. It is a useful exercise to work out all the commutators, and to compare the result with the root diagram.