5 Lie Groups and Lie Algebras

Lie Groups

Now that we have studied several examples of Lie groups, it's time for a definition. A *Lie group* is a group G that is also a smooth manifold. In other words, one can use coordinates to describe the group elements, and one can differentiate with respect to these coordinates. Furthermore, these structures must be compatible, in the sense that the group operations

$$\begin{aligned}(P, Q) &\longmapsto PQ,\\ P &\longmapsto P^{-1},\end{aligned}$$

are smooth maps on G.

We are interested here only in the structure of Lie groups near the identity element. We will therefore usually assume that a Lie group is connected, or equivalently that we are studying the component connected to the identity. All Lie groups can be regarded locally as matrix groups, which we will usually do.

A *representation* of a Lie group G on a vector space V is a group homomorphism

$$\rho : G \longrightarrow \mathrm{End}(V) \tag{97}$$

that takes elements of G to linear maps on V. Thus, a representation of G is an explicit identification of G with certain matrices acting on V. For this reason, the term "representation" is often used to refer to the matrices $\rho(G)$, and occasionally used to refer to V.

Lie Algebras I

The simplest definition of a *Lie algebra* is that it is the tangent space at the identity of a Lie group. This tangent space is a real vector space; thus, Lie algebras are vector spaces. Since Lie groups are locally matrix groups, we can always regard the elements of Lie algebras as matrices. However, all we have so far is the vector space structure, which allows us to add, but not (yet) multiply, Lie algebra elements.

From this point of view, a representation of a Lie algebra on a vector space is simply the result of differentiating a representation of the corresponding Lie group.

Matrix Exponentiation

Given a curve through the identity element of a Lie group, the corresponding Lie algebra element is just the tangent vector to this curve at the identity. How do we go the other way?

The key idea is that there are *nice* curves through the origin, called *1-parameter families* of group elements, with the property that

$$M : \mathbb{R} \longrightarrow G, \tag{98}$$

$$M(0) = 1, \tag{99}$$

$$M(\alpha + \beta) = M(\alpha)M(\beta). \tag{100}$$

In other words, γ is a group homorphism from the additive group of the real numbers into G. Such curves are in 1–1 correspondence with the tangent vectors at the identity. Given a tangent vector $A \in \mathfrak{g}$, how do we find the 1-parameter family $M(\alpha)$ that goes through it, that is, that satisfies

$$A = \dot{M} = M'(0)? \tag{101}$$

Differentiating (100) and using (99) yields the differential equation

$$M'(\alpha) = M(\alpha)A \tag{102}$$

whose solution is

$$M(\alpha) = \exp(A\alpha). \tag{103}$$

But what does it mean to exponentiate a matrix?

We can define matrix exponentials as a power series, so that

$$\exp(A\alpha) = 1 + A\alpha + \frac{1}{2}A^2\alpha^2 + \frac{1}{6}A^3\alpha^3 + ... \tag{104}$$

which turns out to converge for any A. An important special case is when $A^2 = -1$, in which case the series splits into two sums, only one of which involves A. Explicitly, we have

$$A^2 = -1 \Longrightarrow \exp(A\alpha) = \cos\alpha + A\,\sin\alpha \tag{105}$$

where there is of course an implicit identity matrix in the first term. If instead $A^2 = 1$, only the signs change, and we have

$$A^2 = +1 \Longrightarrow \exp(A\alpha) = \cosh\alpha + A\,\sinh\alpha. \tag{106}$$

Finally, if $A^2 = 0$, we have

$$A^2 = 0 \Longrightarrow \exp(A\alpha) = 1 + A\alpha. \tag{107}$$

In practice, even if A itself does not satisfy any of these conditions, it can usually be broken up into blocks that do.

Lie Algebras II

Consider the action of G on itself defined by

$$P \longmapsto MPM^{-1},$$

where $M, P \in G$. If $P = P(\beta)$ is a 1-parameter family, then we can differentiate this action with respect to the parameter, resulting in an action of M on $\dot{P} = P'(0)$. Thus, there is an action of G on its Lie algebra $\mathfrak{g}$, given by

$$X \longmapsto MXM^{-1},$$

where now $X \in \mathfrak{g}$. If we now think of $M = M(\alpha)$ in turn as a 1-parameter family, we can again differentiate, obtaining an action of $\mathfrak{g}$ on itself. But

$$\frac{d}{d\alpha} M(\alpha) X M(\alpha)^{-1} = \frac{dM}{d\alpha} X M(\alpha)^{-1} - M(\alpha) X M(\alpha)^{-2} \frac{dM}{d\alpha} \tag{108}$$

so that

$$\big(M(\alpha) X M(\alpha)^{-1}\big)^{\bullet} = AX - XA = [A, X] \tag{109}$$

where we have used (101). Thus, a Lie algebra always acts on itself by commutators. [3]

We can use this structure to define Lie algebras directly, without starting with a Lie group. A *Lie algebra* is a vector space V, together with an operation

$$\begin{aligned} V \times V &\longrightarrow V \\ (X, Y) &\longmapsto [X, Y] \end{aligned}$$

where the *Lie bracket* $[X, Y]$ is bilinear, antisymmetric, and satisfies the *Jacobi identity*

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0 \tag{110}$$

(which is identically true for matrices).

A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a therefore a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V) \tag{111}$$

that takes elements of $\mathfrak{g}$ to linear maps on V. A representation of $\mathfrak{g}$ is again an explicit identification of $\mathfrak{g}$ with certain matrices acting on V, but in this case the homomorphism preserves commutators. As with Lie groups, the term "representation" is often used to refer to the matrices $\rho(\mathfrak{g})$, and occasionally used to refer to V.

[3] Strictly speaking, we have constructed the commutator as the derivative of a curve *in the Lie algebra*, so it properly lives in the tangent space to the Lie algebra. However, vector spaces are their own tangent spaces, so the result can be regarded as an element of the Lie algebra itself.

6 $\mathbf{SU(2,\mathbb{C}')}$

Representations

By analogy with SU(2), we have

$$\mathrm{SU}(2,\mathbb{C}') = \{M \in \mathbb{C}'^{2\times 2} : M^\dagger M = 1, |M| = 1\}. \tag{112}$$

As with SU(2), we expect there to be 3 degrees of freedom.

Our old friend $M(\phi) \in \mathrm{SO}(2)$ is again in $\mathrm{SU}(2,\mathbb{C}')$, but we give it yet another new name, writing

$$T_y(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{113}$$

Other elements are

$$T_x(\alpha) = \begin{pmatrix} \cosh\alpha & L\sinh\alpha \\ L\sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad T_z(\alpha) = \begin{pmatrix} e^{L\alpha} & 0 \\ 0 & e^{-L\alpha} \end{pmatrix}. \tag{114}$$

Derivatives

We know what to do next: Differentiate! So consider

$$t_y = \dot{T}_y = T_y'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{115}$$

$$t_x = \dot{T}_x = T_x'(0) = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix}, \tag{116}$$

$$t_z = \dot{T}_z = T_z'(0) = \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix}. \tag{117}$$

These matrices are linearly independent, and therefore span the tangent space at the identity.

Let's compute the commutators. The result is

$$[t_x, t_y] = 2t_z, \qquad [t_y, t_z] = 2t_x, \qquad [t_z, t_x] = -2t_y. \tag{118}$$

These commutators are almost, but not quite, those of $\mathfrak{su}(2) = \mathfrak{su}(2,\mathbb{C})$. However, complexifying $\mathfrak{su}(2,\mathbb{C}')$ would yield the same algebra as complexifying $\mathfrak{su}(2,\mathbb{C})$; the signs wouldn't then matter, since they can be changed by multiplication with i. Thus, $\mathfrak{su}(2,\mathbb{C}')$ must be a real form of $\mathfrak{su}(2)$. But which one?

Noting that $t_y = s_y$, $t_x = L\sigma_x$, and $t_z = L\sigma_z$ shows that

$$\mathfrak{su}(2,\mathbb{C}') \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1) \tag{119}$$

since the factors of L do not change the commutators. However, we can no longer use Hermiticity to determine which elements are rotations, and which are boosts, since all elements of $\mathfrak{su}(2,\mathbb{C}')$ are anti-Hermitian.

A more robust mechanism is provided by considering matrix squares. Recall that matrix exponentiation yields ordinary trig functions for matrices that square to minus the identity, and hyperbolic trig functions for matrices that square to the identity. Rotations correspond to ordinary trig, with compact orbits; boosts correspond to hyperbolic trig, with noncompact orbits. The correct generalization of this argument is provided by the *Killing form*

$$B(X,Y) = \mathrm{tr}\,(XY), \tag{120}$$

which yields a norm that is negative for rotations, and positive for boosts, that is

$$\begin{aligned} B(X,X) < 0 &\Longleftrightarrow X \text{ is a rotation,} \\ B(X,X) > 0 &\Longleftrightarrow X \text{ is a boost.} \end{aligned} \tag{121}$$

Using either the Killing form, or simply looking at the trig functions, we conclude that t_y is a rotation, but t_x and t_z are boosts. Comparison with $\mathfrak{sl}(2,\mathbb{R})$ shows that the only difference is that we have multiplied the boosts by L, changing their Hermiticity, but nothing else.

Comparison with SO(2, 1)

We can realize this identification explicitly by considering matrices of the form

$$Y = \begin{pmatrix} -x & z+Lt \\ z-Lt & x \end{pmatrix}. \tag{122}$$

The matrix Y is *tracefree* and *Hermitian*, and we can act on Y with $\mathrm{SU}(2,\mathbb{C}')$ via the action

$$Y \longmapsto MYM^\dagger \tag{123}$$

which preserves both of these conditions. Furthermore, we have

$$|Y| = -(x^2+z^2-t^2), \tag{124}$$

and, since $|M| = 1$,

$$|MYM^\dagger| = |Y| \tag{125}$$

Thus, M can be identified with an element of $\mathrm{SO}(2,1)$, and we have shown that

$$\mathrm{SU}(2,\mathbb{C}') \cong \mathrm{Spin}(2,1). \tag{126}$$

Generalization to $\mathbb{C}' \otimes \mathbb{C}$

Consider now the algebra $\mathbb{C}' \otimes \mathbb{C}$. The tensor product of two algebras consists of linear combinations of formal products. We could write $(1,1)$, $(L,1)$, $(1,i)$ and (L,i) for these products, then define operations by acting on each algebra separately. However, it is easier to drop the parentheses, effectively just multiplying these expressions out; the elements of $\mathbb{C}' \otimes \mathbb{C}$ are linear combinations of 1, L, i, and iL. Since

$$(L,1)(1,i) = (L,i) = (1,i)(L,1) \tag{127}$$

we have

$$iL = Li. \tag{128}$$

As a vector space, $\mathbb{C}' \otimes \mathbb{C}$ is clearly isomorphic to $\mathbb{R}^4$. But the multiplication table is different from the other 4-dimensional algebras we have seen, $\mathbb{H}$ and $\mathbb{H}'$.

Conjugation in a tensor product algebra is defined by conjugating *both* elements, that is

$$\overline{(a,b)} = (\overline{a}, \overline{b}). \tag{129}$$

Thus, although

$$\overline{i} = -i, \qquad \overline{L} = -L, \tag{130}$$

as usual, we have

$$\overline{iL} = iL. \tag{131}$$

So what is $\mathfrak{su}(2, \mathbb{C}' \otimes \mathbb{C})$? Reasoning by analogy with $\mathfrak{su}(2, \mathbb{C}')$, it is clear that a basis for $\mathfrak{su}(2, \mathbb{C}' \otimes \mathbb{C})$ is given by

$$\{L\sigma_x, L\sigma_y, L\sigma_z, -i\sigma_x, -i\sigma_x, -i\sigma_y\}.$$

This basis is the same as the basis for $\mathfrak{sl}(2, \mathbb{C})$ apart from the factors of L, which again appear only in the boosts. These factors change the Hermiticity, but not the commutators, so we conclude that

$$\mathfrak{su}(2, \mathbb{C}' \otimes \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3,1). \tag{132}$$

Thus, the use of $\mathbb{C}'$ makes it easy to identify the boosts—they contain L—and allows boosts to be represented using anti-Hermitian matrices.