4 SO(3,1)

Representations

Rotations exist not only in ordinary Euclidean space, but also in spacetime. However, the notion of "orthogonal" changes. Rather than preserving the Euclidean inner product

$$|v|^2 = v^T v,$$

spacetime rotations preserve the *squared interval*

$$|v|^2 = v^T G v \tag{74}$$

where, in $3+1$ spacetime dimensions,

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{75}$$

The *Lorentzian* inner product G is symmetric ($G^T = G$) and nondegenerate ($|G| \neq 0$), but is *not* positive definite. We say that G has *signature* $(3,1)$ based on the pattern of signs, which is invariant; more generally, the signature of an inner product can consist of any pair of (positive) integers.

The *Lorentz group* consists of transformations that preserve the Lorentzian inner product. By analogy with the Euclidean case, we therefore define

$$\mathrm{SO}(3,1) = \{M \in \mathbb{R}^{4\times 4} : M^T G M = G, |M| = 1\}. \tag{76}$$

What are the elements of $\mathrm{SO}(3,1)$? It is easy to see that any matrix of the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{M} \end{pmatrix} \tag{77}$$

with $\mathcal{M} \in \mathrm{SO}(3)$ will be in $\mathrm{SO}(3,1)$. We will call such elements *rotations*, and continue to label them by their names in $\mathrm{SO}(3)$. In addition, we have the *Lorentz transformations*, also known as *boosts*, between the spatial coordinates and the single timelike coordinate.

Explicitly, we have the three rotations

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \tag{78}$$

$$R_y(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{79}$$

$$R_z(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{80}$$

and the three boosts

$$B_x(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{81}$$

$$B_y(\alpha) = \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\alpha & 0 & \cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{82}$$

$$B_z(\alpha) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}. \tag{83}$$

These six families of (generalized) rotations almost, but not quite, generate SO(3, 1). In even dimensions,

$$|-M| = |M|, \tag{84}$$

so we can multiply these matrices by -1. However, unlike rotations, for which we can achieve the same effect by rotating by an angle of π (in each independent plane), boosts can not be used to change signs. Equivalently, SO(2) is topologically a circle, which is connected, but SO(1, 1) is topologically a hyperbola, which is not. Since we are interested in the tangent space at the identity, for disconnected groups such as SO(3, 1) we will only study the component that is connected to the identity. With this caveat, the six families above can indeed be regarded as generators of SO(3, 1).

Derivatives

The derivatives of these generators at the identity ($\alpha = 0$) form a basis for the Lie algebra $\mathfrak{so}(3,1)$ of SO(3, 1), which is therefore 6-dimensional. The Lie algebra commutators turn out to be

$$[r_x, r_y] = r_z, \quad [b_x, b_y] = -r_z, \quad [r_x, b_y] = b_z, \quad [r_x, b_x] = 0, \tag{85}$$

as well as cyclic permutations of these expressions.

Comparison with SO(3) and SU(2)

We have seen how to represent SO(3) using 2×2 complex matrices, thanks to the local isomorphism of SO(3) with SU(2). Can something similar be done for SO(3, 1)? Yes, indeed.

When acting on complex 2×2 Hermitian matrices, SU(2) preserves both the determinant and the trace. Let's put the trace back in, writing

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}. \tag{86}$$

The matrix X is still Hermitian, but no longer tracefree. What about its determinant? We have

$$|X| = t^2 - (x^2 + y^2 + z^2), \tag{87}$$

which is just (minus) the Lorentzian norm! Thus, transformations that preserve the determinant of X will also preserve this norm—and hence can be identified as elements of SO(3, 1).

As with SU(2), we consider the action

$$X \longmapsto MXM^\dagger$$

so we seek 2×2 complex matrices M that satisfy

$$|MXM^\dagger| = |X|, \tag{88}$$

and we also impose our standard condition that $|M| = 1$. The group of such matrices is the *special linear group* $\mathrm{SL}(2,\mathbb{C})$, that is,

$$\mathrm{SL}(2,\mathbb{C}) = \{M \in \mathbb{C}^{2\times2} : |M| = 1\}. \tag{89}$$

Clearly, we have $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$, but elements of SU(2) also preserve the trace of X, which we no longer require. How many independent elements ("generators") of $\mathrm{SL}(2,\mathbb{C})$ do we expect? A 2×2 complex matrix has $4 \times 2 = 8$ real degrees of freedom; specifying the determinant is 1 complex constraint, or 2 real constraints. Thus, in addition to the 3 generators $\{S_m\}$ of SU(2), we expect 3 additional generators. What are they? It is not hard to see that these generators are

$$\Sigma_x(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \tag{90}$$

$$\Sigma_y(\alpha) = \begin{pmatrix} \cosh\alpha & -i\sinh\alpha \\ i\sinh\alpha & \cosh\alpha \end{pmatrix}, \tag{91}$$

$$\Sigma_z(\alpha) = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \tag{92}$$

and further that these generators can be identified with boosts in SO(3, 1).

We have shown that SO(3, 1) is locally the same as $\mathrm{SL}(2,\mathbb{C})$, but, as with SO(3) and SU(2), this equivalence is not global. Rather, we have

$$\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C}), \tag{93}$$

where Spin(3, 1) is the double cover of SO(3, 1).

What happens at the Lie algebra level? Since $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$, we know that the s_m, our basis elements for $\mathfrak{su}(2)$, are in $\mathfrak{sl}(2,\mathbb{C})$. What about the boosts? The derivatives of the Σ_m (at the identity) are precisely the Pauli matrices σ_m! Thus, the s_m and σ_m together form a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which is 6-dimensional, and we have verified explicitly that

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1) \tag{94}$$

Wait a minute. Shouldn't $\mathfrak{sl}(2,\mathbb{C})$ be a *complex* vector space?

The answer to this question depends on the context. Lie *groups* are normally treated as having real parameters, so the tagent space at the origin is always a *real* vector space. However, when classifying Lie algebras it is customary to complexify this vector space, for reasons we will discuss later. Since

$$\sigma_m = i s_m, \tag{95}$$

$\mathfrak{sl}(2,\mathbb{C})$ can be viewed as *either* a real 6-dimensional Lie algebra *or* a complex 3-dimensional Lie algebra.

However, that's not the whole story. As a complex algebra, $\mathfrak{sl}(2,\mathbb{C})$ is really just the complexification of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, that is

$$\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C} \otimes \mathfrak{su}(2). \tag{96}$$

But $\mathfrak{sl}(2,\mathbb{C})$ is also the complexification of its real subalgebra, $\mathfrak{sl}(2,\mathbb{R})$—which turns out to be the same as $\mathfrak{so}(2,1)$.

Be warned that "$\mathfrak{so}(3)$" is often used to denote both the real Lie algebra $\mathfrak{so}(3) = \mathfrak{so}(3,\mathbb{R})$ and its complexification $\mathfrak{so}(3,\mathbb{C}) = \mathbb{C} \otimes \mathfrak{so}(3,\mathbb{R})$, and that other notations are also used. Real subalgebras of $\mathfrak{so}(3,\mathbb{C})$ whose complexification is again all of $\mathfrak{so}(3,\mathbb{C})$ are called *real forms* of $\mathfrak{so}(3)$. Real forms can be classified by the number of boosts they contain; the real forms of $\mathfrak{so}(3)$ are $\mathfrak{so}(3)$, also called the *compact* real form (since SO(3) is compact), with 0 boosts, and $\mathfrak{so}(2,1)$, with 2 boosts.

It is a worthwhile exercise to explicitly compare the bases of the real forms $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{sl}(2,\mathbb{C})$, noting the relationship between rotations and boosts.