3 SU(2)

Representations

Just as orthogonal groups preserve the magnitude of *real* vectors, the *unitary* groups preserve the magnitude of *complex* vectors. The "rotation" group in two complex dimensions is known as SU(2). If $v \in \mathbb{C}^2$, its magnitude is defined by

$$|v|^2 = v^\dagger v \tag{42}$$

where † denotes the conjugate transpose, that is

$$v^\dagger = \overline{v}^T. \tag{43}$$

The magnitude of a complex vector is still a non-negative real number, just as for real vectors.

So if we want to preserve the magnitude of v, we must have

$$(Mv)^\dagger(Mv) = (v^\dagger M^\dagger)(Mv) = v^\dagger v \tag{44}$$

or simply

$$M^\dagger M = 1. \tag{45}$$

Matrices satisfying (45) are called *unitary* matrices; the "U" in SU(2) stands for unitary. As before, the "S" stands for special, and refers to the additional condition that

$$|M| = 1. \tag{46}$$

Thus, we have

$$\mathrm{SU}(2) = \{M \in \mathbb{C}^{2\times 2} : M^\dagger M = 1, |M| = 1\}. \tag{47}$$

It's not that difficult to find the most general matrix satisfying (45) and (46), but let's first ask how many degrees of freedom there are. A 2×2 complex matrix has 8 real degrees of freedom, the matrix equation (45) imposes 4 constraints, and the determinant condition (46) imposes one more constraint. Thus, we expect 3 parameters in our general element—just as for SO(3).

Let's start with some examples. Our old friend $M(\phi) \in \mathrm{SO}(2)$ is in SU(2), but we give it a new name, writing

$$S_y(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{48}$$

Other examples of unitary matrices are

$$S_x(\alpha) = \begin{pmatrix} \cos\alpha & -i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}, \qquad S_z(\alpha) = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}. \tag{49}$$

Derivatives

We know what to do next: Differentiate! So consider

$$s_y = \dot{S}_y = S_y'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{50}$$

$$s_x = \dot{S}_x = S_x'(0) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{51}$$

$$s_z = \dot{S}_z = S_z'(0) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{52}$$

These matrices are linearly independent, and therefore span the tangent space at the identity.

So what? Well, let's compute the commutators. The result is

$$[s_x, s_y] = 2s_z, \qquad [s_y, s_z] = 2s_x, \qquad [s_z, s_x] = 2s_y. \tag{53}$$

In other words, the commutators resulting from SU(2) are the same [1] as those resulting from SO(3). So the infinitesimal versions of these two groups are the same. So they're the same group—at least locally.

Comparison with SO(3)

We can realize this identification explicitly by considering matrices of the form

$$X = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}. \tag{54}$$

The matrix X is *tracefree*, that is

$$\mathrm{tr}\,(X) = 0, \tag{55}$$

and *Hermitian*, that is,

$$X^\dagger = X. \tag{56}$$

We can act on X with SU(2) via the action

$$X \longmapsto MXM^\dagger \tag{57}$$

which preserves both of these conditions. Furthermore, we have

$$|X| = -(x^2 + y^2 + z^2), \tag{58}$$

and, since $|M| = 1$,

$$|MXM^\dagger| = |X| \tag{59}$$

Thus, M can be identified with an element of SO(3)!

[1] The factor of 2 can easily be eliminated by dividing each of our basis elements s_m by 2, or equivalently by replacing α by $\alpha/2$ in S_m.

However, that's not the whole story. Since there are 2 Ms in our action, we can not tell the actions of $\pm M$ apart. Thus, SU(2) is the double cover of SO(3). These two groups have different global structure, but their local structure is the same.

Every orthogonal group SO(n) admits such a double cover, but SO(3) is the only one whose double cover is a unitary group. The double cover of SO(n) is called Spin(n); we have shown that

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2). \tag{60}$$

But our interest is in the local structure, and SU(2) is the same as SO(3) locally.

Properties

You may recognize the s_m as being i times the corresponding *Pauli matrices*, that is,

$$s_m = -i\sigma_m. \tag{61}$$

What are the properties of the Pauli matrices? They are *Hermitian*, they are *tracefree*, and they square to the identity, that is

$$\sigma_m^\dagger = \sigma_m, \tag{62}$$

$$\mathrm{tr}\,(\sigma_m) = 0, \tag{63}$$

$$\sigma_m^2 = 1. \tag{64}$$

Similarly, the s_m are *anti-Hermitian*, tracefree, and square to minus the identity, that is

$$s_m^\dagger = -s_m, \tag{65}$$

$$\mathrm{tr}\,(s_m) = 0, \tag{66}$$

$$s_m^2 = -1. \tag{67}$$

More generally, if $M(\alpha)$ is any family of elements of SU(2) such that $M(0) = 1$, then $M(\alpha)^\dagger M(\alpha) = 1$ by (45). Differentiating this equation, we obtain

$$M'(\alpha)^\dagger M(\alpha) + M(\alpha)^\dagger M'(\alpha) = 0 \tag{68}$$

and evaluating this result at $\alpha = 0$ yields

$$A^\dagger + A = 0 \tag{69}$$

where we have written A for $\dot{M} = M'(0)$. This argument is true quite generally: *If the elements of a Lie group are unitary matrices, then the elements of the corresponding Lie algebra are anti-Hermtian.* Thus, we can describe the Lie algebra $\mathfrak{su}(2)$ of SU(2) as

$$\mathfrak{su}(2) = \{A \in \mathbb{C}^{2\times 2} : A^\dagger + A = 0, \mathrm{tr}\,(A) = 0\}. \tag{70}$$

The same argument shows that infinitesimal orthogonal matrices are antisymmetric, as you can verify in the case of SO(3) and $\mathfrak{so}(3)$. Thus,

$$\mathrm{SO}(3) = \{M \in \mathbb{R}^{3\times 3} : M^T M = 1, |M| = 1\}, \tag{71}$$

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} : A^T + A = 0, \mathrm{tr}\,(A) = 0\} \tag{72}$$

and we have shown that

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \tag{73}$$

not merely as vector spaces, but as Lie algebras (since commutators are preserved).