Ricci Curvature

Let $\{e_m\}$ be a basis for $\mathfrak{g}$

$E_m = \mathrm{ad}\, e_m \Rightarrow \{E_m\}$ is a basis for $\mathrm{ad}\,\mathfrak{g}$

Denote the dual bases by $\{e_*^m\}$, $\{E_*^m\}$

$$\Rightarrow B(X, Y) = \mathrm{tr}(\mathrm{ad}X\ \mathrm{ad}Y)$$
$$= E_*^m(\mathrm{ad}X\ \mathrm{ad}Y\ E_m)$$
$$= E_*^m(\mathrm{ad}X\ \mathrm{ad}[Y, e_m])$$
$$= E_*^m(\mathrm{ad}[X, [Y, e_m]])$$
$$= e_*^m([X, [Y, e_m]])$$

$$\therefore \boxed{B(X, Y) = e_*^m([X, [Y, e_m]])}$$

& recall that $B(Y, X) = B(X, Y)$

Ricci curvature is defined by

$$\boxed{Ric(X, Y) = -e_*^m(R(X, e_m)Y)}$$
$$= +\tfrac{1}{4} e_*^m([[X, e_m], Y])$$
$$= -\tfrac{1}{4} e_*^m([Y, [X, e_m]])$$
$$= -\tfrac{1}{4} B(Y, X) = \underline{-\tfrac{1}{4} B(X, Y)}$$

Einstein metric!

Roots

Step 1: Cartan subalgebra

Idea: choose a maximal, nondegenerate abelian subalgebra

commuting

diagonalizable (ing)

Fact (hard): (finite, semisimple)
All Cartan subalgebras are conjugate $(h_2 = Q h_1 Q^{-1})$

can get from one to another by change of basis

Ex: $su(2)$: $\frac{1}{2}\sigma_z \longleftrightarrow L_z$
$su(3)$: λ_3, λ_8

$\Rightarrow \exists$ eigenbasis for g

$\Rightarrow g = h \oplus$ eigenspaces with (some) nonzero eigenvalues

Ex: $su(3) = \langle \lambda_3, \lambda_8 \rangle \oplus \langle L_+ \rangle \oplus \cdots \oplus \langle N_- \rangle$

$\langle \lambda_3, \lambda_8 \rangle = h$

eigenspaces of $\underline{\underline{h}}$

Ex: $[\lambda_3, M_-] = M_-$

$[\lambda_8, M_-] = \sqrt{3}\, M_-$

$\therefore H \in h \Rightarrow H = a\lambda_3 + b\lambda_8$

$$\Rightarrow [H, M_-] = a[\lambda_3, M_-] + b[\lambda_8\, M_-]$$
$$= (a + b\sqrt{3})\, M_-$$

different eigenvalue

same eigenspace

Let $\alpha : h \to \mathbb{R}$
$H \mapsto a + b\sqrt{3}$

so $\alpha \in h^*$

then $[H, M_-] = \alpha(H)\, M_- \quad \forall H \in h$

Idea: $\exists$ such an $\alpha \in h^*$ for each eigenspace

g_α

$$\therefore \boxed{g = h \oplus_{\alpha \in R} g_\alpha}$$

where $H \in h,\ X \in g_\alpha \Rightarrow [H, X] = \alpha(H) X$

The α are called roots of g

$(h = g_0$ but normally assume $\alpha \neq 0)$

$\alpha(H)$ gives the α^{th} eigenvalue of

$$H = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

Properties of Roots

① $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $H \in \mathfrak{h}$

$$\Rightarrow [H, X] = \alpha(H) X$$
$$[H, Y] = \beta(H) Y$$

$$\Rightarrow [H, [X, Y]] = [[H, X], Y] + [X, [H, Y]]$$
$$= (\alpha(H) + \beta(H)) [X, Y].$$

$$\therefore \boxed{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}}$$

could be zero vector space

same argument shows

$$\boxed{\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}}$$

② $B([H, X], Y) + B(X, [H, Y]) = 0$

$$\Rightarrow (\alpha(H) + \beta(H)) B(X, Y) = 0$$

$$\therefore \boxed{\alpha + \beta \neq 0 \Rightarrow \mathfrak{g}_\alpha \perp \mathfrak{g}_\beta}$$

$\Longrightarrow \mathfrak{g}_\alpha \perp \mathfrak{g}_\alpha$!

This argument works unchanged if $\alpha = 0$:

$$\therefore \boxed{\mathfrak{g}_\alpha \perp \mathfrak{h}}$$

③ $H, K \in h \quad X \in g_\alpha$

$$\Rightarrow [H,[K,X]] = [K,[H,X]]$$
$$= \alpha(H)\alpha(K) X$$
$$\Rightarrow B(H,K) = \sum_\alpha \alpha(H)\alpha(K)$$ (nondegenerate!)

(obvious if written out in terms of diagonal matrices!)

④ Let $\{e_m\}$ be a basis for h
& $\lambda_m = \alpha(e_m)$

$\therefore H \in h \Rightarrow H = h^m e_m$
$$\Rightarrow \alpha(H) = h^m \lambda_m$$

$\alpha \neq 0 \Rightarrow$ can not be all of h

$\therefore$ if $\lambda_m \neq 0$ can always choose h^m so that $\alpha(H) = 0$

$\therefore$ Ker(α) is $n-1$ dimensional

in g_α
in $g_{-\alpha}$

⑤ $H \in \text{Ker}\,\alpha$, $K \in h_\alpha \Rightarrow \alpha(H) = 0$ & $K = [X, Y]$
$$\Rightarrow B(H,K) = B(H,[X,Y]) = B([H,X],Y)$$
$$= \alpha(H) B(X,Y) = 0$$
$$\Rightarrow \text{Ker}\,\alpha < h_\alpha^\perp \Rightarrow \text{Ker}\,\alpha = h_\alpha^\perp$$

$$\therefore \boxed{h_\alpha \text{ is 1-dimensional}}$$

⑥ $0 \neq X \in g_\alpha \Rightarrow \exists Y \in g_{-\alpha} : B(X, Y) \neq 0$

nondegeneracy

$\therefore$ let $2H_\alpha = [X, Y] \in h_\alpha$

$$\Rightarrow [H_\alpha, X] = \alpha(H_\alpha) X \ \& \ [H_\alpha, Y] = -\alpha(H_\alpha) Y$$

⑦ $\alpha \neq 0 \Rightarrow \exists H \in h : \alpha(H) \neq 0$

$$\Rightarrow B(H, 2H_\alpha) = B(H, [X, Y]) = B([H, X], Y)$$

$$= \alpha(H) B(X, Y) \neq 0$$

$$\Rightarrow \underline{H_\alpha \neq 0}$$

⑧ Suppose $\underline{\alpha(H_\alpha) = 0}$

& let V be <u>any</u> representation of h

Consider the eigenspace

$$V_\lambda = \{ v \in V : \rho(H_\alpha) v = \lambda v \}$$

for some <u>fixed</u> λ

since X, Y commute with H_α by assumption, they act on V_λ

$$\therefore [\rho(X), \rho(Y)] v = 2 \rho(H_\alpha) v = \lambda v$$

so $[\rho(X), \rho(Y)] = \lambda \mathbb{I}$ on V_λ

But $\mathrm{tr}(LHS) = 0 \Rightarrow \lambda = 0$

$\Rightarrow$ the <u>only</u> eigenvalue of H_α is 0

$$\Rightarrow \beta(H_\alpha) = 0 \ \forall \text{ roots } \beta \Rightarrow H_\alpha = 0 \ \boxed{\times}$$

$$\therefore \boxed{\alpha(H_\alpha) \neq 0}$$

$\therefore$ Have $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, $H_\alpha = [X, Y]$
with $\alpha(H_\alpha) \neq 0$

$$\Rightarrow \{X, Y, H\} \longleftrightarrow \{L_+, L_-, L_z\}$$

$\therefore$ suppose $Z \in \mathfrak{g}_\alpha$

$$\Rightarrow [Y, Z] \in \mathfrak{h}_\alpha \subset \mathfrak{h}$$

$$\Rightarrow [Y, Z] = a H_\alpha \qquad \text{1-dimensional!}$$
$$= -a[Y, X]$$

$\Rightarrow$ Z is in the <u>same</u> 1-dimensional "ladder" as X

$$\Rightarrow Z = -a X$$

$$\Rightarrow \boxed{\mathfrak{g}_\alpha \text{ is 1-dimensional}}$$

$X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ determined only up to scale

but H_α fixed by $\boxed{\alpha(H_\alpha) = 1}$

Two-Dimensional Root Diagrams

Recall : $\frac{\alpha \cdot \beta}{\alpha \cdot \alpha} \in \frac{1}{2}\mathbb{Z} \Rightarrow 4\cos^2\theta = 0, 1, 2, 3, 4$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 0$

Goal construct all 2d root diagrams

Need : α, β indpt $\Rightarrow \theta \neq 0$

Case 1 : $\theta = \frac{\pi}{2} \Rightarrow \alpha \perp \beta \quad \mapsto d_2$

Case 2 : $\theta = \frac{\pi}{3} \Rightarrow mn = 1 \Rightarrow |\beta| = |\alpha|$

$\mapsto a_2$

Case 3 : $\theta = \frac{\pi}{4} \Rightarrow mn = 2$

$\Rightarrow |\beta| = \sqrt{2}\,|\alpha| \quad \mapsto b_2$

or

$|\beta| = \frac{1}{\sqrt{2}}|\alpha| \quad \mapsto c_2$

(same in 2d, but not in higher dim)

Case 4 : $\theta = \frac{\pi}{6} \Rightarrow mn = 3$

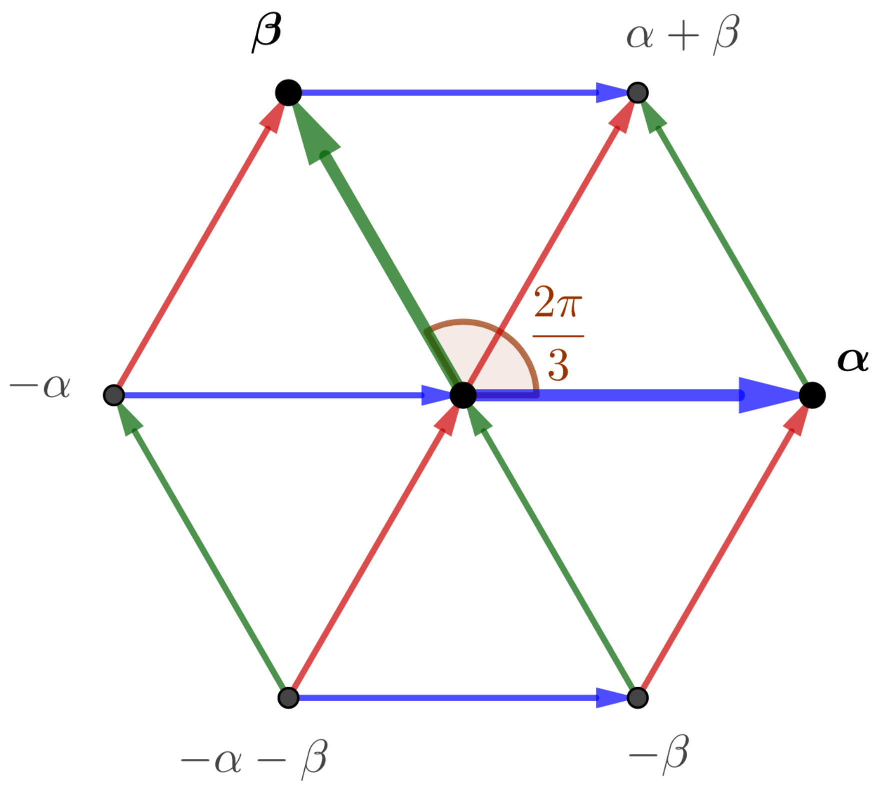
$\Rightarrow |\beta| = \sqrt{3}\,|\alpha|$

or

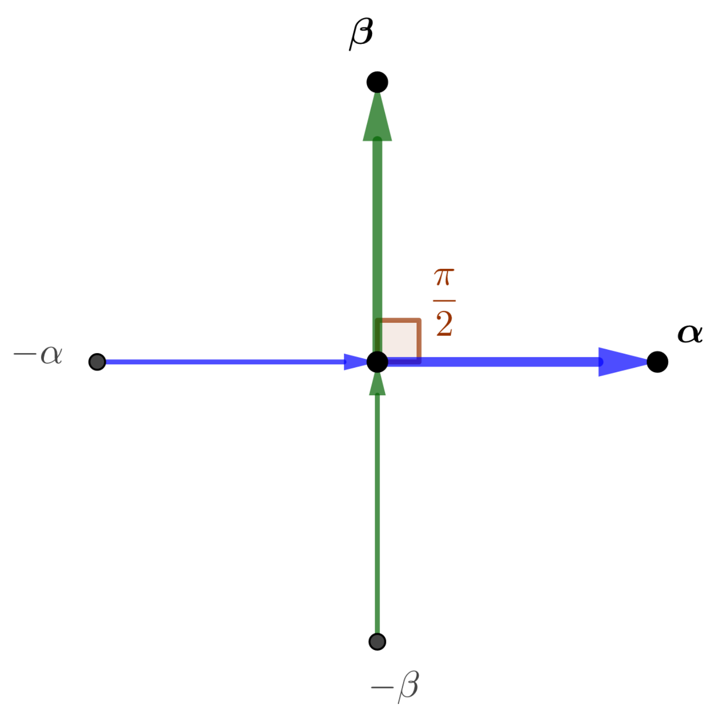
$|\beta| = \frac{1}{\sqrt{3}}|\alpha|$

$\mapsto g_2$ (only example)

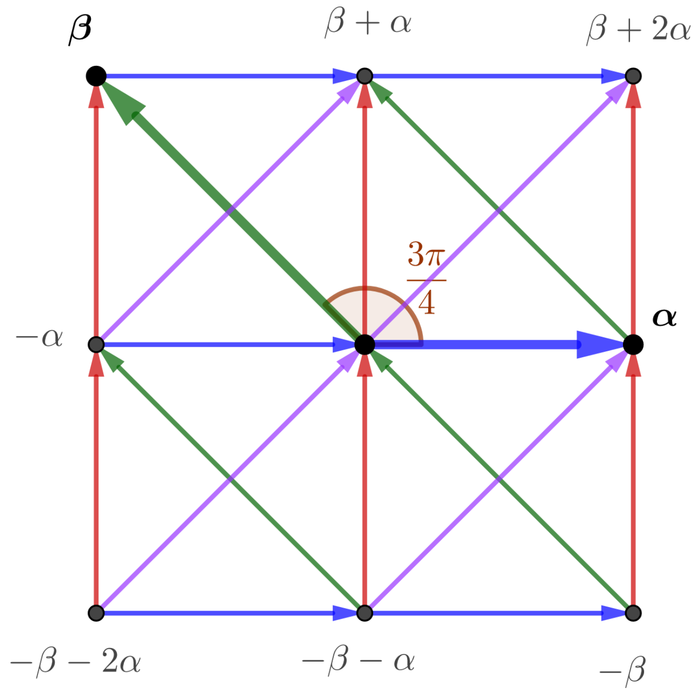
$$\boxed{a_2 \cong su(3)}$$



$$\boxed{d_2 \cong so(4) \cong su(2) \oplus su(2)}$$

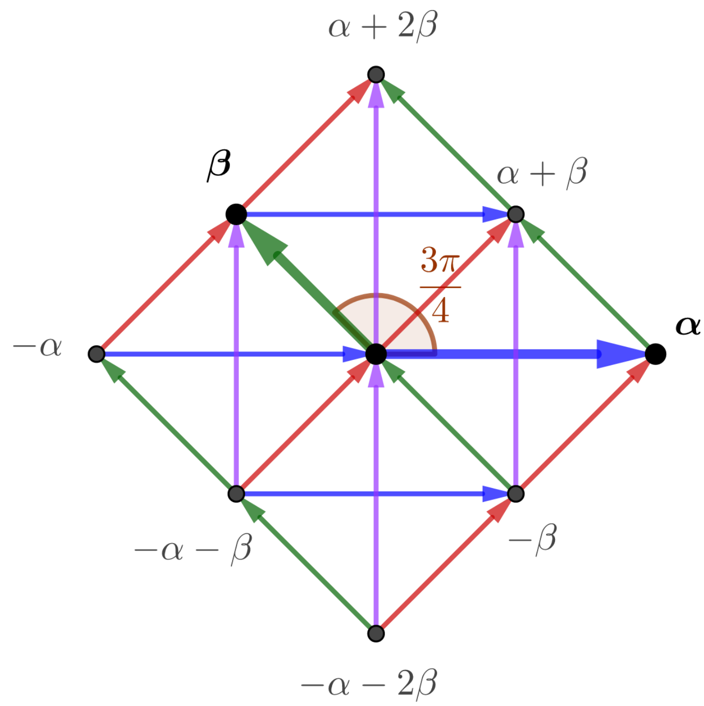


$$\boxed{b_2 \cong so(5)}$$

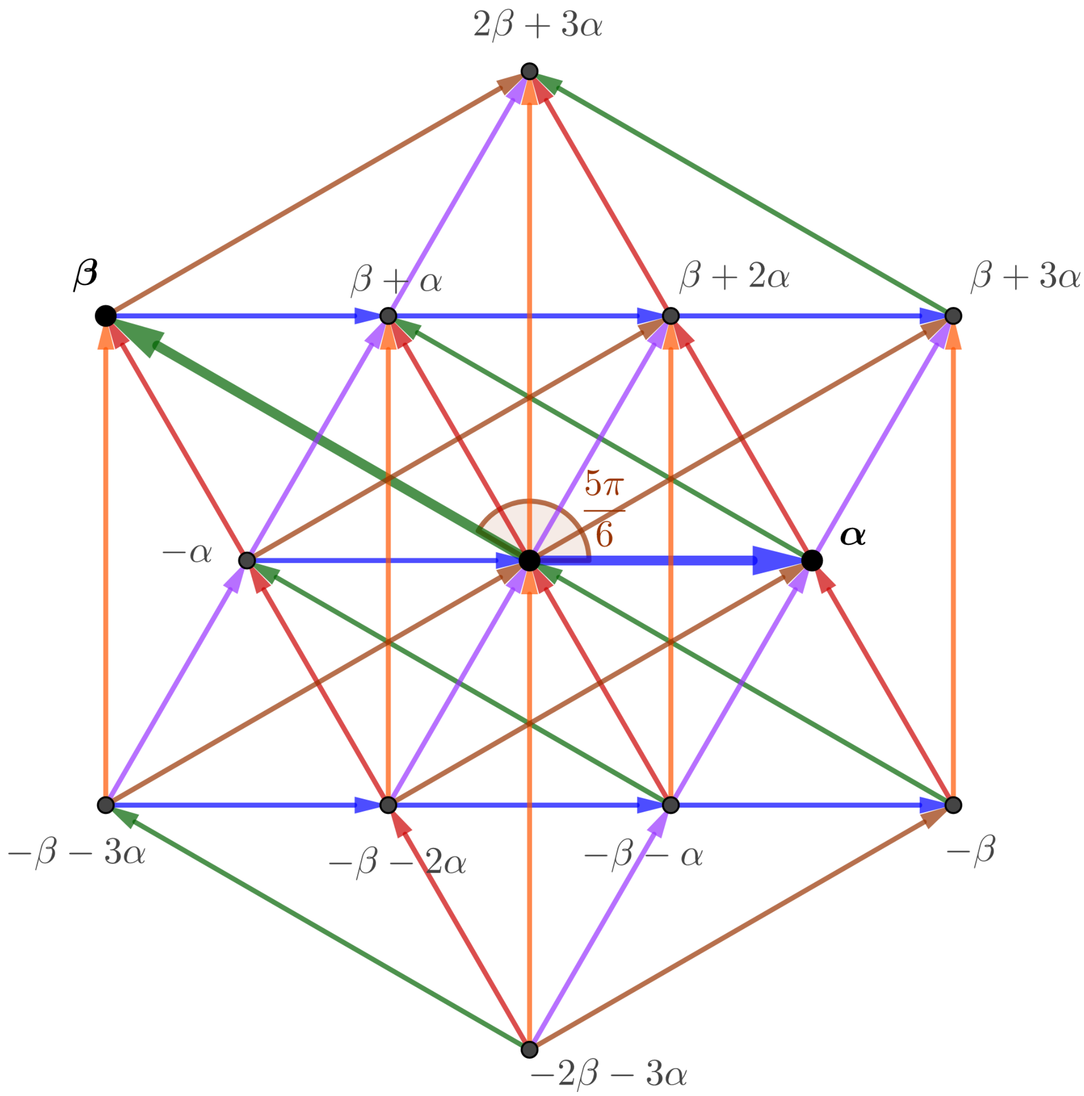


$$\boxed{c_2 \cong su(2,\mathbb{H}) \cong so(5)} \cong sp(2)$$

"symplectic"



$\mathfrak{g}_2$



$2\beta + 3\alpha$
$\boldsymbol{\beta}$
$\beta + \alpha$
$\beta + 2\alpha$
$\beta + 3\alpha$
$\frac{5\pi}{6}$
$-\alpha$
$\boldsymbol{\alpha}$
$-\beta - 3\alpha$
$-\beta - 2\alpha$
$-\beta - \alpha$
$-\beta$
$-2\beta - 3\alpha$

<u>Lemma</u> : $\boxed{\begin{array}{l}\alpha \neq \beta \text{ simple} \\ \quad \Longrightarrow \ \alpha \cdot \beta \leq 0\end{array}}$

<u>Pf</u> : $\alpha \cdot \beta = 0$ is allowed, so

assume $\alpha \cdot \beta \neq 0$

$$\Longrightarrow r_\alpha(\beta) = \beta - 2\frac{\alpha \cdot \beta}{\alpha \cdot \alpha}\alpha = \beta - m\alpha$$

is a root, with $m \neq 0$

$\Longrightarrow \beta, \ \beta - \frac{m}{|m|}\alpha, \ \ldots, \beta - m\alpha$ are roots

<u>Suppose $m > 0$</u> $\Longrightarrow \beta - \alpha$ is a root

$\Longrightarrow \alpha - \beta$ is a root

$\Longrightarrow$ one of $\beta - \alpha, \alpha - \beta$ is positive

But $\alpha = (\alpha - \beta) + \beta$
$\beta = (\beta - \alpha) + \alpha$ so α, β not simple

$\boxed{\divideontimes}$

$\therefore \ m < 0 \Longrightarrow \alpha \cdot \beta < 0$ ✓

Dynkin Diagrams

Recall:

- Roots come in pairs $\pm\alpha$
- Each $g_\alpha \cup g_{-\alpha} \cup [g_\alpha, g_{-\alpha}] \cong su(2)$
- $su(2)$ reps $\Rightarrow$ reflection symmetry

$$\Rightarrow 2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha} \in \mathbb{Z}$$

$$\Rightarrow 4\cos^2\theta \in \{0,1,2,3,4\}$$

- $\exists$ basis of simple roots α_i

(a) α_i indpt $\Rightarrow$ $\cos^2\theta \neq 1$

(b) α_i, α_j simple $\Rightarrow$ $\alpha_i \cdot \alpha_j \leq 0$

$\therefore$ exactly 4 cases

$4\cos^2\theta_{ij}$	θ_{ij}	ratio
0	$\pi/2$	$-$
1	$\pi/3$	1
2	$\pi/4$	$\sqrt{2}$
3	$\pi/6$	$\sqrt{3}$

Dynkin diagram:

- a dot for each root
- $4\cos^2\theta$ lines connecting them
- indicate larger root if necessary

(I) $\exists$ at most $K-1$ connections between K simple roots

(A connection is any number of lines)

$4\cos^2\theta$	$2\cos\theta$	# of lines	connections
0	0	0	0
1	-1	1	1
2	$-\sqrt{2}$	2	1
3	$-\sqrt{3}$	3	1

$\Rightarrow 2\cos\theta \leq -\#$ of connections

Pf: Let $\alpha = \sum_{i=1}^{K} \frac{\alpha_i}{|\alpha_i|}$

$\Rightarrow \quad \alpha \neq 0$ (α_i are a basis)

$$\Rightarrow 0 < \alpha \cdot \alpha = \sum_{i,j} \frac{\alpha_i \cdot \alpha_j}{|\alpha_i||\alpha_j|}$$

$$= K + \sum_{i<j} 2\cos\theta_{ij}$$

$$\leq K - \#\text{ of connections}$$

$\Rightarrow \nexists$ closed loops in Dynkin diagrams!

(II) $\exists \le 3$ lines at each point

Pf: Suppose β is connected to k simple roots β_i

- no cycles $\Rightarrow$ none of the β_i are connected

$$\Rightarrow \{\beta_i\} \text{ orthogonal}$$

Use Gram-Schmidt to construct orthogonal basis $\{\beta_0, \beta_i\}$ of $\langle \beta, \beta_i \rangle$

β independent $\Rightarrow$ $\beta \cdot \beta_0 \neq 0$

$$\Rightarrow |\beta|^2 = \sum_0^k \frac{(\beta \cdot \beta_i)^2}{\beta_i \cdot \beta_i} > \sum_1^k \frac{(\beta \cdot \beta_i)^2}{|\beta_i|^2}$$

$$\Rightarrow \# \text{ of lines} = \sum 4 \cos^2 \theta_i$$

$$= \sum \frac{(\beta \cdot \beta_i)^2}{|\beta|^2 |\beta_i|^2} < 4 \quad \checkmark$$

(III) Simple chains can be replaced by a single root

(A simple chain consists of single lines)

Suppose: 

$$\Rightarrow \frac{\alpha_i \cdot \alpha_{i+1}}{\alpha_i \cdot \alpha_i} = -\frac{1}{2} \quad \& \quad \alpha_i \cdot \alpha_i = \alpha_{i+1} \cdot \alpha_{i+1} = Q^2$$

Let $\alpha = \sum_1^K \alpha_i$

$$\Rightarrow \alpha \cdot \alpha = \sum_{i,j} \alpha_i \cdot \alpha_j = \sum_{i=1}^{K} \alpha_i \cdot \alpha_i + \sum_{i=1}^{K-1} 2\alpha_i \cdot \alpha_{i+1}$$

$$= KQ^2 - (K-1)Q^2 = Q^2$$

Furthermore, any other root β is connected to at most one α_i, and then $\alpha \cdot \beta = \alpha_i \cdot \beta$

Thus, replacing $\{\alpha_i\}$ by α yields a valid diagram ✓

Allowed Diagrams

($\mathfrak{g}$ simple $\Rightarrow$ diagram connected)

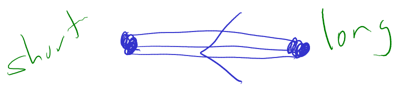
Type A — a_n

n roots in a simple chain

Type G — g_2

2 roots connected by 3 lines

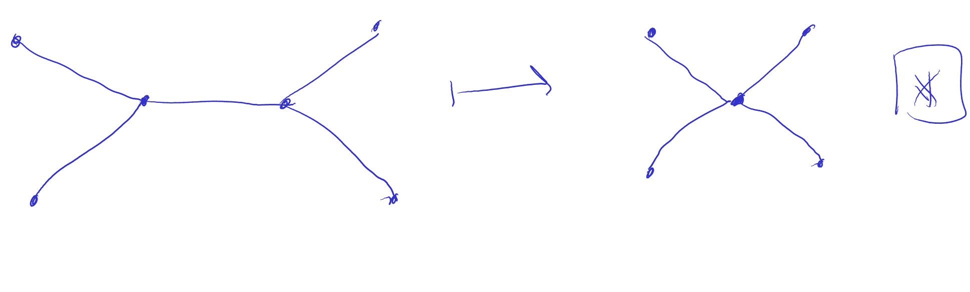
(can not extend!)



Remaining cases

- Can not have 2 double lines:

- Can not have 2 branch points



Simple Chains

If $\alpha_1 \ldots \alpha_p$ is a simple chain,

then $\alpha_m \cdot \alpha_m = \underline{p}^2$

$$\alpha_m \cdot \alpha_{m+1} = -\frac{1}{2}\underline{p}^2$$

Let $\alpha = \sum_1^p m \frac{\alpha_m}{|\alpha_m|}$

$$\Longrightarrow \alpha \cdot \alpha = \sum_{m=1}^{p} m^2 \frac{\alpha_m \cdot \alpha_m}{|\alpha_m|^2} + 2\sum_{m=1}^{p-1} m(m+1) \frac{\alpha_m \cdot \alpha_{m+1}}{|\alpha_m||\alpha_{m+1}|}$$

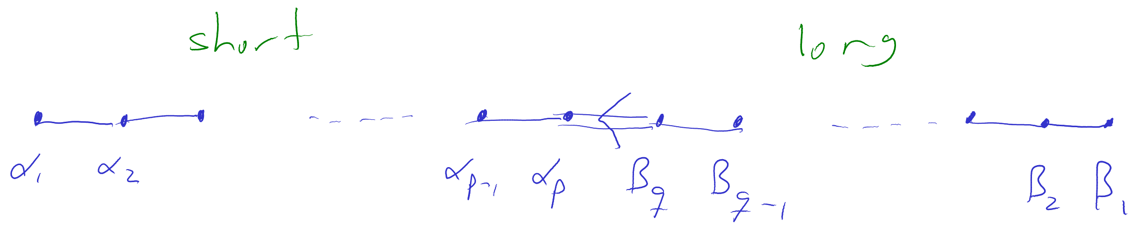
$$= \sum_{m=1}^{p} m^2 + 2\sum_{m=1}^{p-1}\left(-\tfrac{1}{2}\right)m(m+1)$$

$$= p^2 + \sum_{m=1}^{p-1}\left(m^3 - m(m+1)\right)$$

$$= p^2 - \sum_{m=1}^{p-1} m = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$$

$$\therefore \boxed{\alpha \cdot \alpha = \frac{p(p+1)}{2}}$$

Double Roots



$$\alpha = \sum_m \frac{\alpha_m}{|\alpha_m|}, \quad \beta = \sum_k \frac{\beta_k}{|\beta_k|}$$

$$\Longrightarrow \quad \alpha_m \cdot \alpha_m = P^2, \quad \beta_k \cdot \beta_k = Q^2 = 2P^2$$

$$\alpha \cdot \alpha = \frac{p(p+1)}{2}, \quad \beta \cdot \beta = \frac{q(q+1)}{2}$$

$$4(\alpha_p \cdot \beta_q)^2 = 2P^2Q^2$$

Furthermore, $\alpha \cdot \beta = p\frac{\alpha_p}{P} \cdot q\frac{\alpha_q}{Q} = pq\frac{\alpha_p \cdot \beta_q}{PQ}$

$$\Rightarrow (\alpha\cdot\beta)^2 = \frac{p^2q^2}{2}$$

$$< |\alpha|^2|\beta|^2$$

$\cos^2\theta \neq 1$

$$= \frac{p(p+1)q(q+1)}{4}$$

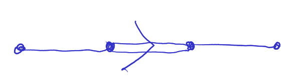
$$\therefore \quad 2pq < (p+1)(q+1)$$

$$\Rightarrow \quad pq < p+q+1 \Rightarrow \boxed{(p-1)(q-1) < 2}$$

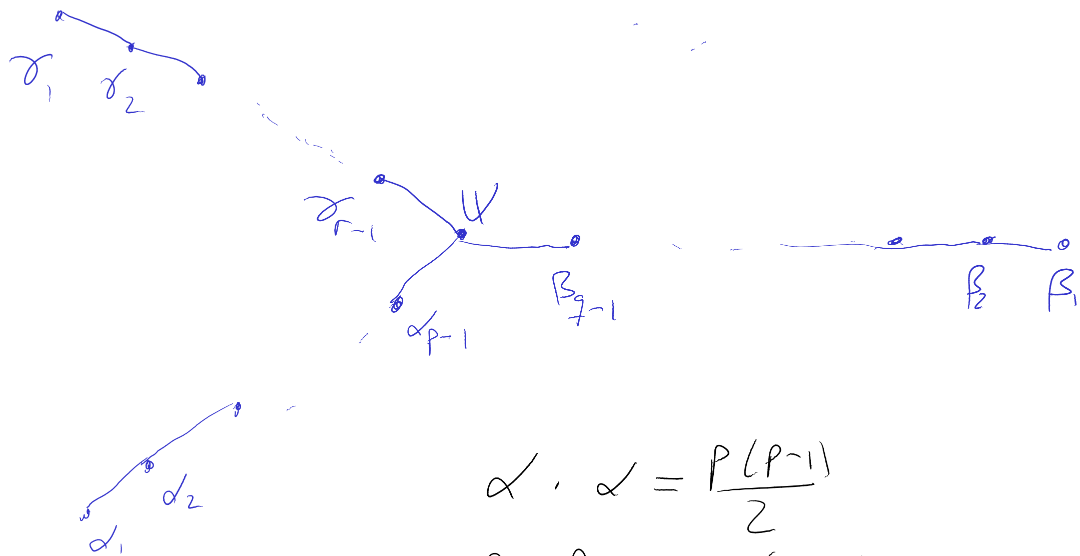
$\therefore$ $p = 1$ Type B b_{p+1}

or $q = 1$ Type C c_{q+1}

or $p = q = 2$ Type F f_4



Branch Points



$$\alpha \cdot \alpha = \frac{p(p-1)}{2}$$

$$\beta \cdot \beta = \frac{q(q-1)}{2}$$

$$\gamma \cdot \gamma = \frac{r(r-1)}{2}$$

Furthermore,

$$(\alpha \cdot \psi)^2 = \frac{(p-1)^2(\alpha_{p-1} \cdot \psi)^2}{\alpha_{p-1} \cdot \alpha_{p-1}} = (p-1)^2 |\psi|^2 \frac{1}{4}$$

$\cos \frac{2\pi}{3}$

$$\Longrightarrow \frac{(\alpha \cdot \psi)^2}{|\alpha|^2 |\psi|^2} = \frac{p-1}{2p} = \frac{1}{2}\left(1 - \frac{1}{p}\right)$$

But $\alpha_m, \beta_k, \gamma_\ell$ orthogonal $\Rightarrow \alpha, \beta, \gamma$ orthogonal

& ψ indpt of $\alpha_m, \beta_k, \gamma_\ell \Rightarrow$ indpt of α, β, γ

$$\therefore \quad \psi = \frac{\psi \cdot \alpha}{\alpha \cdot \alpha}\alpha + \frac{\psi \cdot \beta}{\beta \cdot \beta}\beta + \frac{\psi \cdot \gamma}{\gamma \cdot \gamma}\gamma + \frac{\psi \cdot \psi_0}{\psi_0 \cdot \psi_0}\psi_0$$

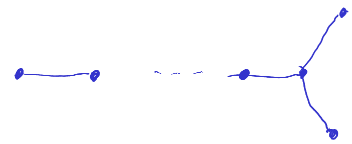
nonzero

$$\Longrightarrow \psi \cdot \psi > \left(\frac{\psi \cdot \alpha}{\alpha \cdot \alpha}\right)^2 + \left(\frac{\psi \cdot \beta}{\beta \cdot \beta}\right)^2 + \left(\frac{\psi \cdot \gamma}{\gamma \cdot \gamma}\right)^2$$

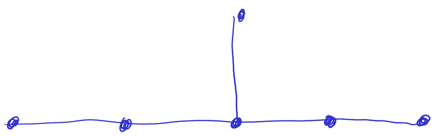
$$\Longrightarrow \frac{3}{2} - \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < 1 \Longrightarrow \boxed{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1}$$

$$\boxed{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1}$$

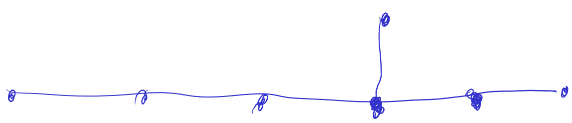
$\therefore$ $\underline{q=r=2}$ $\underline{\text{Type D}}$ d_{p+2}



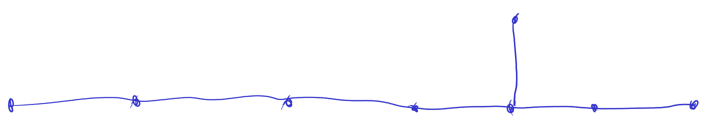
or $\underline{r=2,\ q=3,\ p=3,4,5}$ $\underline{\text{Type E}}$



e_6



e_7



e_8