Complex Numbers

- Tell me something you know about complex numbers

$z = x + iy = re^{i\phi}$ $\qquad \bar{z} = x - iy = re^{-i\phi}$

$\mathbb{C}$ is a vector space ← what dimension?!

normed : $|z|^2 = z\bar{z} = x^2 + y^2 = r^2$

algebra : $\exists$ multiplication

nondegenerate : $|z| = 0 \Rightarrow z = 0$ ← careful! correct def for positive-definite case only

composition : $|ab| = |a||b|$

division : $z \neq 0 \Rightarrow z^{-1} = \dfrac{\bar{z}}{|z|^2}$

- Arms

$1, i, -i$

$3i$

$e^{\frac{i\pi}{6}}$

z (your choice)

$e^{\frac{i\pi}{2}} z$

iz

$\bar{z}$

Rotations

- Tell me something you know about rotations in 2 dimensions

composition (multiplication)
can undo (inverses)

$\longmapsto$ group!

Representations

① angle $\phi \in S^1$

② matrix $M(\phi) \in \mathbb{R}^{2\times 2}$

$$M(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

③ complex number $w(\phi) \in \mathbb{C}$

$$w(\phi) = e^{i\phi}$$

Conditions: $v \in \mathbb{R}^2$, $|v|^2 = v^T v$, $|Mv| = |v|$

$$\Rightarrow \underline{M^T M = \mathbb{1}} \qquad (\& \det M = 1)$$

$$|wz| = |z| \Rightarrow \underline{|w| = 1}$$

Isomorphism.

$$e^{i\phi} = \cos\phi + i\sin\phi \longleftrightarrow M(\phi) = \cos\phi\,\mathbb{1} + \sin\phi\,\Omega$$

$$x + iy \longleftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Exponentiation

- what does $e^{k\alpha}$ mean?

$$e^{k\alpha} = 1 + k\alpha + \frac{(k\alpha)^2}{2} + \dots$$

unique soln of $\frac{df}{d\alpha} = kf$, $f'(0) = k$

But $M'(\phi) = \Omega M$ & $\Omega = M'(0)$ $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$w'(\phi) = i w$ & $i = w'(0)$

$\Downarrow$

$\Omega^2 = -\mathbb{1}$!

$\Rightarrow$ $w = e^{i\phi}$ ✓

$M = e^{\Omega\phi}$

Orthogonal Groups

$v \in \mathbb{R}^n, \quad |v|^2 = v^T v, \quad M \in \mathbb{R}^{2\times 2}$

want : $|Mv| = |v|$

$$\Rightarrow \quad (Mv)^T(Mv) = v^T v$$
$$\overset{\shortparallel}{v^T M^T M v}$$

$$\Rightarrow \quad \boxed{M^T M = \mathbb{1}} \quad \leftarrow O(n)$$

$$\boxed{SO(n) = \{ M \in \mathbb{R}^{n\times n} : M^T M = \mathbb{1} \ \& \ \det M = 1 \}}$$

group! manifold! $\longmapsto$ Lie group

Derivatives

Suppose $M(\theta) \in SO(n)$, $M(0) = \mathbb{1}$

$$\Rightarrow \quad M(\theta)^T M(\theta) = \mathbb{1}$$

$$\Rightarrow \quad M'(\theta)^T M(\theta) + M(\theta)^T M'(\theta) = 0$$

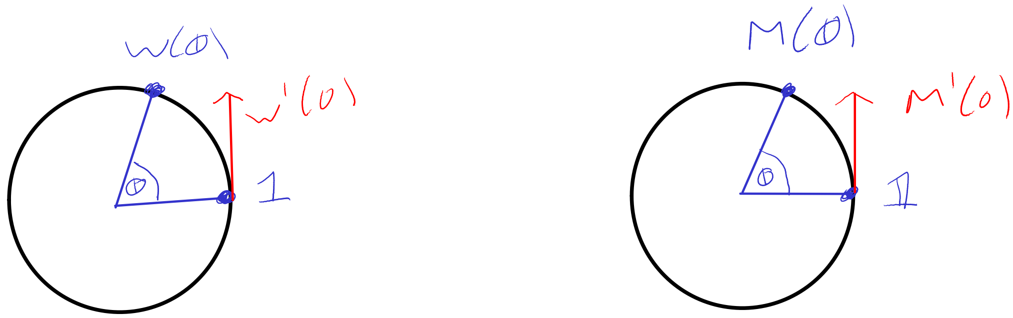
$$\Rightarrow \quad M'(0)^T \mathbb{1} + \mathbb{1}^T M'(0) = 0$$

$$\longmapsto \quad \boxed{so(n) := \{ A \in \mathbb{R}^{n\times n} : A^T + A = 0 \}}$$

vector space! algebra! $\longmapsto$ Lie algebra

closed under commutator

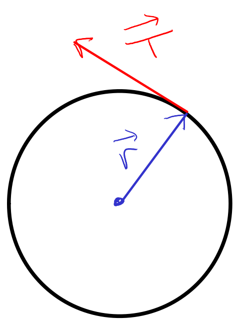
SO(2)



Idea: Lie group = group with smooth parameters
= group + manifold

Lie algebra = tangent vectors at identity

Vector fields



$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{r}(\theta) = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\Rightarrow \vec{T}(\theta) = \frac{d\vec{r}}{d\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$= -y\hat{x} + x\hat{y}$$

Differential geometry

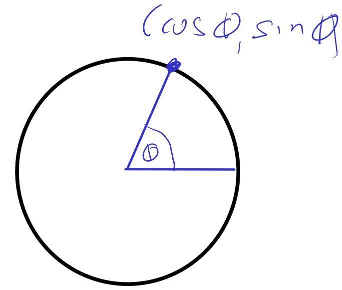
vector fields are directional derivative operators

$$\vec{v}(f) = \vec{v}\cdot\vec{\nabla}f$$

$$\therefore \vec{T}(f) = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$

$$\Rightarrow \vec{T} \longleftrightarrow -y\,\partial_x + x\,\partial_y = \partial_\theta$$

$$SO(2) \longleftrightarrow M(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = e^{\Omega\phi}$$



$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega^2 = -\mathbb{I}$$
$$= M'(0)$$

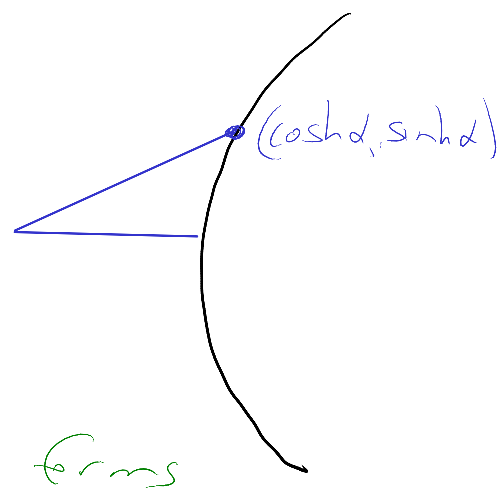
$$SO(1,1) \longleftrightarrow P(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$
$$P'(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Lambda, \quad \Lambda^2 = +\mathbb{I}$$
$$\Rightarrow P(\alpha) = e^{\Lambda\alpha}$$

Signature

M preserves $v^T v = v^T \mathbb{I} v$

P preserves $v^T \eta v$

with $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



"$so(1,1)$ and $so(2)$ are real forms of (complex) $so(2)$"

Split complex numbers

"hyperbolic numbers"

$$L^2 = +1$$

$$\mathbb{C}' = \mathbb{R} \oplus \mathbb{R}L \qquad \text{compare: } \mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$\Rightarrow e^{L\alpha} = \cosh\alpha + L\sinh\alpha \longleftrightarrow P(\alpha)$$

Rotations in 3d

- Tell me something you know about rotations in 3d

  not commutative

  group

Representations

I $\quad R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$

$\mapsto$ Euler angles

$$R(\theta, \phi, \psi) = R_z(\phi) R_y(\theta) R_z(\psi)$$
$$= [\text{mess}]$$

Derivatives $\qquad r_m = \dot{R}_m = R_m'(0)$

II
$$\underline{Z} = x\partial_y - y\partial_x$$
$$\underline{X} = y\partial_z - z\partial_y$$
$$\underline{Y} = z\partial_x - x\partial_z$$

Topology of $SO(3)$

Idea: Euler angles $\theta, \phi \longleftrightarrow S^2$

$\psi \longleftrightarrow S^1$

$\therefore SO(3) \approx S^2 \times S^1$

Not quite: correct locally but not globally

$\longmapsto$ Hopf bundle: $\begin{array}{c} S^3 \\ \downarrow S^1 \\ S^2 \end{array}$

so perhaps $SO(3) \approx S^3$

Not quite: rotations around antipodal points equivalent

$\therefore SO(3)$ is topologically $S^3/\mathbb{Z}_2$

so $\boxed{SO(3) \cong \mathbb{RP}^3}$

Commutators

Goal: need product for Lie algebra

Matrices: $\boxed{[A,B] = AB - BA}$

$\Rightarrow [\Gamma_x, \Gamma_y] = \Gamma_z, \quad [\Gamma_y, \Gamma_z] = \Gamma_x, \quad [\Gamma_z, \Gamma_x] = \Gamma_y$

$\Rightarrow$ so(3) closed under commutation ✓

Vector Fields

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

$$\Rightarrow [X, Y] = -Z$$

conventional sign, can be eliminated by change of basis

Properties: bilinear
antisymmetric
(Jacobi identity)

Check: (exercise) $A^T = -A, \; B^T = -B$

$\Rightarrow [A,B]^T = -[A,B]$

$\Rightarrow$ so(n) closes under commutation

Definitions

A **Lie group** is a group that is also a smooth manifold (surface), on which the group operations are smooth

multiplication and inverses

A **Lie algebra** is a vector space that is closed under a bilinear operation $[\,,\,]$ with:

$$[Y,X] = -[X,Y] \qquad \text{antisymmetry}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Jacobi identity

(automatic for matrices!)

Fact: elements of Lie group near $\mathbb{I}$ look like $M(\alpha) = e^{A\alpha}$ for A in Lie algebra

matrix exponentiation if matrix Lie group

$\Rightarrow$ $M(\alpha)$ is a 1-parameter family:

$$M(0) = \mathbb{I}$$

$$M(\alpha+\beta) = M(\alpha)M(\beta)$$

$$M(-\alpha) = M(\alpha)^{-1}$$

Vector Fields

Day 03

Recall $\boxed{\vec{v}(f) = \vec{v} \cdot \vec{\nabla} f}$

Ex: $\hat{x}(f) = \dfrac{\partial f}{\partial x}$ so $\hat{x} \longleftrightarrow \dfrac{\partial}{\partial x} = \partial_x$

$$(x\hat{y} - y\hat{x})(f) = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi}$$

Claim $[\vec{v}, \vec{w}]$ is a vector field

Idea

$$v_x \hat{x}\left(v_y \hat{y}(f)\right) - v_y \hat{y}\left(v_x \hat{x}(f)\right)$$

$$= v_x \frac{\partial}{\partial x}\left(v_y \frac{\partial f}{\partial y}\right) - v_y \frac{\partial}{\partial y}\left(v_x \frac{\partial f}{\partial x}\right)$$

$$= v_x \frac{\partial v_y}{\partial x}\frac{\partial f}{\partial y} + \cancel{v_x v_y \frac{\partial^2 f}{\partial x \partial y}} - v_y \frac{\partial v_x}{\partial y}\frac{\partial f}{\partial x} - \cancel{v_y v_x \frac{\partial^2 f}{\partial y \partial x}}$$

$$= \left(v_x \frac{\partial v_y}{\partial x}\partial_y - v_y \frac{\partial v_x}{\partial y}\partial_x\right) f$$

$$\Rightarrow [v_x \hat{x}, v_y \hat{y}] = v_x \frac{\partial v_y}{\partial x}\hat{y} - v_y \frac{\partial v_x}{\partial y}\hat{x}$$

Rotations

$$\boxed{\begin{aligned} \mathcal{Z} &= x\partial_y - y\partial_x \\ \mathcal{X} &= y\partial_z - z\partial_y \\ \mathcal{Y} &= z\partial_x - x\partial_y \end{aligned}}$$

$$\Rightarrow [\mathcal{X}, \mathcal{Y}] = -\mathcal{Z} \ldots$$

Jacobi Identity

Recall:

$$\boxed{[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0}$$

Rearrange:

① $$\boxed{[Z,[X,Y]] = [[Z,X],Y] + [X,[Z,Y]]}$$

product rule!
(Lie bracket is a derivation)

② $$\boxed{[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]}$$

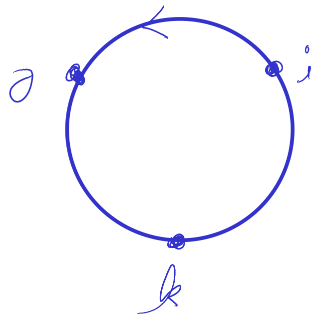
homomorphism!
(action of commutator is commutator of actions)

Quaternions

$\mathbb{H}$ for Hamilton

$$q \in \mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$$i^2 = j^2 = k^2 = -1$$



$$ij = k = -ji, \ldots$$

- Show pictures of Broome Bridge (1853)
- $q \longleftrightarrow$ Hamilton
$\vec{v} \longleftrightarrow$ Gibbs   late 1800s
- Modern applications: robotics, aeronautics, computer graphics

Correspondence

$$\vec{v} = x\hat{x} + y\hat{y} + z\hat{z} \longleftrightarrow v = xi + yj + zk \in \mathrm{Im}\,\mathbb{H}$$

$$\text{"}-\vec{v}\cdot\vec{w} + \vec{v}\times\vec{w}\text{"} \longleftrightarrow vw$$

quaternionic product incorporates both dot & cross products

Quaternionic Rotations

$q \in \mathbb{H} \Rightarrow q = a + bj$ with $a, b \in \mathbb{C}$

SWBQ: Find a, b in terms of w, x, y, z with $\boxed{q = w + ix + jy + kz}$

KA: Working with a partner, represent

1

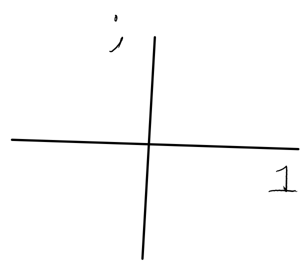
k

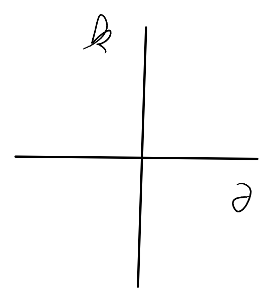
$i + j$

q (your choice)

iq, $e^{i\alpha} q$

qi, $q e^{i\alpha}$

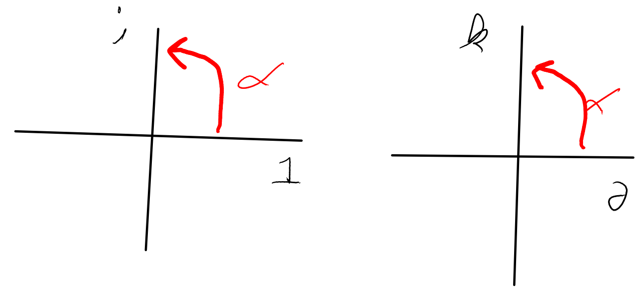




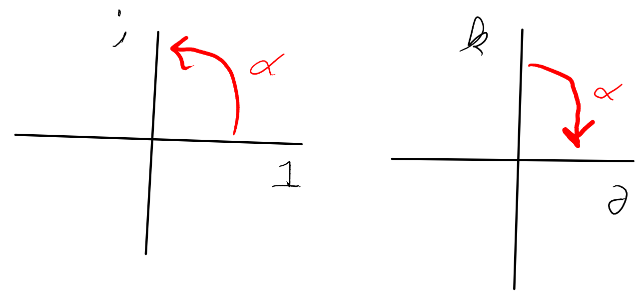
Idea: $iq = i(a+bj) = ia + ibj$

$qi = (a+bj)i = ia - ibj$

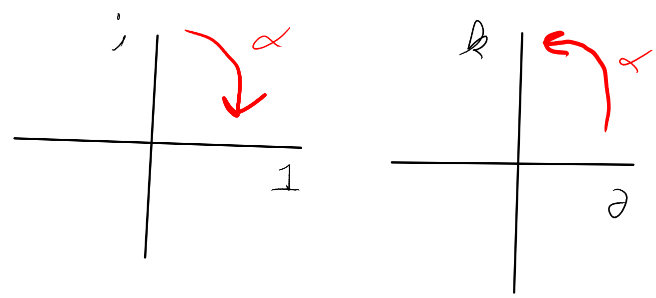
$\therefore\ e^{i\alpha}q$:



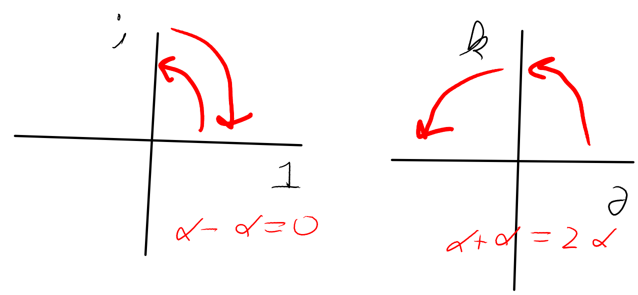
$qe^{i\alpha}$:



$qe^{-i\alpha}$:



$e^{i\alpha}qe^{-i\alpha}$:



$\therefore\ e^{\frac{i\alpha}{2}} q\, e^{-\frac{i\alpha}{2}} \longleftrightarrow$ rotation by α in jk plane

"about i axis in $\operatorname{Im}\mathbb{H}$"

Can replace i by any unit element $u \in \operatorname{Im}\mathbb{H}$, $|u|=1$

$$\therefore\ \boxed{e^{\frac{u\alpha}{2}} q\, e^{-\frac{u\alpha}{2}} \longleftrightarrow \text{rotation in } \operatorname{Im}\mathbb{H} \text{ by } \alpha \text{ about } u \text{ axis}}$$

Day 04

Homomorphism

Given: Lie algebra g with Lie bracket
$$[X, Y]_g$$

$$\Rightarrow \quad g \longrightarrow Lin(g, g) \longleftrightarrow \mathbb{R}^{n\times n}$$
$$(n = \dim g)$$
$$X \overset{L}{\longmapsto} L_X$$
where $L_X : g \longrightarrow g$
$$Y \longmapsto [X, Y]$$

$P, Q \in Lin(g, g) \Longrightarrow$
$[P, Q]_L \in Lin(g, g)$ where
$$[P, Q]_L : Z \longmapsto P(Q(Z)) - Q(P(Z))$$

Recall: Jacobi identity $\Longrightarrow$
$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]$$

$$\Longleftrightarrow \quad L_{[X, Y]_g} = [L_X, L_Y]_L$$

$\therefore$ L is a Lie algebra homomorphism

Unitary Matrices

Recall: $SO(n) = \{M \in \mathbb{R}^{n\times n} : M^T M = \mathbb{1},\ \det M = 1\}$

preserves $|v|^2 = v^T v$

Generalization: $SU(n) = \{M \in \mathbb{C}^{n\times n} : M^\dagger M = \mathbb{1},\ \det M = 1\}$

preserves $|v|^2 = v^\dagger v$

Ex: $SU(2)$

$$S_y(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$S_x(\alpha) = \begin{pmatrix} \cos\alpha & -i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}$$

$$S_z(\alpha) = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{+i\alpha} \end{pmatrix}$$

1-parameter families

Differentiate!

$$S_m = \dot{S}_m = S_m'(0)$$

$$\Rightarrow \quad S_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$M^\dagger M = \mathbb{1} \Rightarrow \dot{M}^\dagger + M = 0 \Rightarrow M^\dagger = -M$$

$$M = e^{A\alpha}$$

$$= \mathbb{1} + A\alpha + A^2\alpha^2 + \cdots$$

suppose A is diagonalizable

$$\Rightarrow A = P D P^{-1}$$

$$\Rightarrow M = P e^{D\alpha} P^{-1}$$

$$\Rightarrow \det M = \det e^{D\alpha}$$

$$= e^{d_1\alpha} e^{d_2\alpha} \cdots e^{d_n\alpha}$$

$$= e^{(d_1 + d_2 + \cdots d_n)\alpha}$$

$$= e^{\mathrm{tr} A\alpha}$$

True in general:

$$\boxed{\det e^{A} = e^{\mathrm{tr} A}}$$

$$\therefore \det M = 1$$

$$\Rightarrow \mathrm{tr}\, \dot{M} = 0$$

Commutators

$$[S_x, S_y] = 2S_z$$

$$[S_y, S_z] = 2S_x$$

$$[S_z, S_x] = 2S_y$$

same as $SO(3)$!

$(S_m \longmapsto S_m/2)$

$\therefore$ $\boxed{su(2) \cong so(3)}$

Comparison with $SO(3)$

$$H_3^0(\mathbb{C}) = \{ \underline{P} \in \mathbb{C}^{2\times 2} ; \underline{P}^\dagger = \underline{P} , \operatorname{tr} \underline{P} = 0 \}$$

$$\underline{P} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \in V$$

$$\Longrightarrow M\underline{P}M^\dagger \in H_3^0(\mathbb{R}) \qquad \text{for } M \in SU(2)$$

But $\det \underline{P} = -(x^2+y^2+z^2)$

$\Longrightarrow$ M preserves Euclidean inner product on $\mathbb{R}^3$

$\therefore$ $\underline{P} \longmapsto M\underline{P}M^\dagger$ is in $SO(3)$!

But $\pm M$ induce the same rotation

$\therefore$ $SU(2)$ is the double cover of $SO(3)$

$$\therefore \quad SO(3) \approx \mathbb{RP}^3$$
$$SU(2) \approx \mathbb{S}^3$$

$SO(3,1)$

$$\eta = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad v \in \mathbb{R}^4 \longleftrightarrow \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$|v|^2 = v^T \eta v \longleftrightarrow -t^2 + x^2 + y^2 + z^2$$

$$\Rightarrow \quad SO(3,1) = \{ M \in \mathbb{R}^4 : |Mv| = v \ \& \ \det M = 1 \}$$

$$\Rightarrow \quad M^T \eta M = \eta$$

Clearly, $SO(3) \subset SO(3,1)$

$$M \longmapsto \left(\begin{array}{c|c} 1 & \\ \hline & M \end{array} \right)$$

But also 3 <u>boosts</u>

$$B_x(\alpha) = \left(\begin{array}{cc|c} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & \\ \hline & 0 & \mathbb{1} \end{array} \right)$$

<u>Warning</u> : $\det(-M) = \det M$

$\Rightarrow SO(3,1)$ <u>disconnected</u>

Idea: O vs) (

$SL(2,\mathbb{C})$

$$H_3(\mathbb{C}) = \{Q \in \mathbb{C}^{2\times 2} : Q^\dagger = Q\}$$

$$\Rightarrow Q = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$\Rightarrow \det Q = -(-t^2+x^2+y^2+z^2)$$

$$SL(2,\mathbb{C}) = \{M \in \mathbb{C}^{2\times 2} : \det M = 1\}$$

$$M \in SL(2,\mathbb{C}) \Rightarrow \det MQM^\dagger = \det M$$

$\Rightarrow$ $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$

(really of its component connected to $\mathbb{1}$)

$sl(2,\mathbb{C})$ spanned by

S_m

$$\sigma_m = i S_m$$

Pauli matrices

Day 05

Components

Let $\{e_m\}$ be a basis for Lie algebra $\mathfrak{g}$

$$\Rightarrow \boxed{[e_m, e_n] = C^a{}_{mn}\, e_a}$$

structure constants for $\mathfrak{g}$

$$\Rightarrow [[e_m, e_n], e_p] = C^a{}_{mn}\, C^b{}_{ap}\, e_b$$

|| (by Jacobi identity)

$$\Rightarrow [e_m, [e_n, e_p]] - [e_n, [e_m, e_p]]$$

$$= \left(C^b{}_{mq}\, C^q{}_{np} - C^b{}_{nq}\, C^q{}_{mp} \right) e_b$$

$\therefore$ the matrices $C_m = \left(C_m{}^b{}_q \right)$ satisfy

$$\boxed{C_m C_n - C_n C_m = C^a{}_{mn}\, C_a}$$

and $\therefore$ reproduce the Lie algebra!
(since structure constants are the same)

"adjoint representation" $e_m \mapsto C_m$

Representations

I math

Lie alg

linear endomorphisms

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

vector space

$$[X, Y] \longmapsto \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

Lie algebra homomorphism

matrix multiplication ("composition")

Ex: adjoint: $V = \mathfrak{g}$ "$\mathfrak{g}$ acting on itself"

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$$
$$X \longmapsto \mathrm{ad}_X$$

$$\boxed{\mathrm{ad}_X(Y) = [X, Y]}$$

$$\Rightarrow \mathrm{ad}_{[X,Y]} Z = [[X,Y], Z]$$
$$= [X, [Y, Z]] - [Y, [X, Z]]$$
$$= \mathrm{ad}_X \mathrm{ad}_Y Z - \mathrm{ad}_Y \mathrm{ad}_X Z$$

$$\Rightarrow \boxed{\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]}$$

II physics representation $\longleftrightarrow$ matrices $\rho(\mathfrak{g})$

III particle physics & representation thy: representation $\longleftrightarrow$ vector space V

Killing Form

$\exists$ natural inner product on $\mathfrak{g}$!

$$\boxed{B(X,Y) = \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y)}$$

In practice, $\underbrace{B = \operatorname{tr}(XY)}_{\text{usually}}$ (unique up to overall scale)

Properties:

- symmetric

  $B(Y,X) = B(X,Y)$

- bilinear

Def: $\mathfrak{g}$ semisimple $\iff$ B nondegenerate

$\mathfrak{g}$ simple $\iff$ no proper ideals

$\nexists\, 0 \neq \mathfrak{h} \subsetneq \mathfrak{g} : [\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}$

Fact: $\mathfrak{g}$ semisimple $\iff$ $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with $\mathfrak{g}_m$ simple

Ex: $so(2)$ $\quad \operatorname{ad}_X Y = [X,Y]$

but $[X,Y] \equiv 0$!

$\Longrightarrow \operatorname{ad}_X = 0$

$\Longrightarrow B \equiv 0$ $\quad$ degenerate!

Basis $\quad e_p$

$$\Rightarrow \quad ad_{e_p}(e_q) = [e_p, e_q] = C^a{}_{pq} e_a$$

$$\Rightarrow \quad e_q \longmapsto C^a{}_{pq} e_a$$

$$\Rightarrow \quad \begin{array}{c} \\ q \\ \\ \end{array} \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \longmapsto \begin{pmatrix} C^1{}_{pq} \\ \vdots \\ C^n{}_{pq} \end{pmatrix}$$

$$\Rightarrow \quad ad_{e_p} = \begin{array}{c} \begin{pmatrix} & C^1{}_{pq} & \\ & \vdots & \\ & C^n{}_{pq} & \end{pmatrix} \\ q \end{array} = \left(C^a{}_{pq} \right) = C_p$$

$$\therefore \; B(e_p, e_q) = tr(C_p C_q)$$
$$= tr\left(C^a{}_{pk} C^k{}_{qb}\right) = C^m{}_{pk} C^k{}_{qm}$$

Similar argument shows

$$\boxed{B([X,Y],Z) = B([Y,Z],X)}$$

$$\Rightarrow \quad B([Z,X],Y) + B(X,[Z,Y]) = 0$$

$\Rightarrow$ B invariant under G!

G acts on $\mathfrak{g}$: $X \longmapsto MXM^{-1}$

Differentiate : $X \longmapsto AX - XA = [A,X]$

$$\Rightarrow \quad B(X,Y) \longmapsto 0$$

i.e. B unchanged by M

$$\Rightarrow \quad \underline{B(MXM^{-1}, MYM^{-1}) = B(X,Y)}$$

Example $so(5,1)$

- what size matrices are in $so(5,1)$?
- what is the dimension of $so(5,1)$?
- what size is matrix representation of Killing form?
- How many boosts & how many rotations?

Example: $so(4)$

Basis 1: $\Gamma_{xy}, \Gamma_{yz}, \Gamma_{zx}, \Gamma_{wz}, \Gamma_{wx}, \Gamma_{wy}$

Basis 2:

$$\Gamma_{xy} \pm \Gamma_{wz}$$
$$\Gamma_{yz} \pm \Gamma_{wx}$$
$$\Gamma_{zx} \pm \Gamma_{wy}$$

$$\boxed{so(4) = su(2) \oplus su(2)}$$

not simple!!
(only such example)

Day 06

Representations of su(2)

Idea:
- Start with σ_z ← real eigenvalues!
- Find eigenvectors ← in (complex) su(2)!

Recall: $[S_x, S_y] = 2S_z \ \ldots$

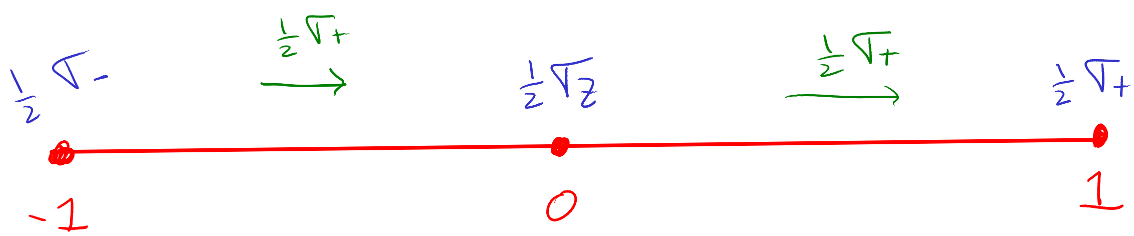
$$\Longrightarrow \left[\tfrac{1}{2}S_x, \tfrac{1}{2}S_y\right] = 2\frac{S_z}{4} = \frac{1}{2}S_z \ \ldots$$

Furthermore, S_z is anti-Hermitian $\longmapsto$ imaginary eigenvalues

$\therefore$ consider $\frac{1}{2}\sigma_z = \frac{i}{2}S_z \in su(2)\otimes\mathbb{C}$

& $\frac{1}{2}(\underbrace{\sigma_x \pm i\sigma_y}_{\sigma_\pm}) = \frac{1}{2}(\sigma_x \mp S_y)$

$$\Longrightarrow \boxed{\begin{aligned} \left[\tfrac{1}{2}\sigma_z, \tfrac{1}{2}\sigma_\pm\right] &= \pm\tfrac{1}{2}\sigma_\pm \\ \left[\tfrac{1}{2}\sigma_+, \tfrac{1}{2}\sigma_-\right] &= 2\left(\tfrac{1}{2}\sigma_z\right) \end{aligned}}$$



Adjoint rep!
Root diagram!

$\frac{1}{2}\sigma_\pm$ are raising/lowering operators

$\{\sigma_z, \sigma_x, S_y\}$ span $sl(2,\mathbb{R})$!
(real form of su(2))

Other Representations

$$\rho : su(2) \longmapsto gl(V)$$

$$L_z = \rho\left(\tfrac{1}{2}\sigma_z\right)$$

$$L_\pm = \rho\left(\tfrac{1}{2}\sigma_\pm\right)$$

$$\Rightarrow \boxed{\begin{aligned} [L_z, L_\pm] &= \pm L_\pm \\ [L_+, L_-] &= 2L_z \end{aligned}}$$

Suppose L_z diagonalizable (true if semi-simple)

Choose basis of eigenvectors $\{w, \dots\}$

$$\Rightarrow L_z w = \lambda w$$

$$\begin{aligned} \Rightarrow L_z L_\pm w &= [L_z, L_\pm] w + L_\pm L_z w \\ &= \pm L_\pm w + \lambda L_\pm w \\ &= (\lambda \pm 1) L_\pm w \end{aligned}$$

For V to be <u>irreducible</u> (no proper subreps)

$\Rightarrow$ basis must be $\{L_\pm^n w\}$

For V to be <u>finite</u>, must have <u>largest</u> eigenvalue

$$\mapsto \quad L_+ w = 0 \qquad (\lambda \text{ largest})$$

$$\Rightarrow \quad L_+L_-w = [L_+, L_-]w + L_-\cancel{L_+w}$$
$$= 2L_z w = 2\lambda w$$

$$\Rightarrow \quad L_+L_-L_-w = [L_+, L_-]L_-w + L_-L_+L_-w$$
$$= 2L_z L_-w + 2\lambda L_-w$$
$$= 2(2\lambda - 1)L_-w$$

$$\vdots$$

$$\Rightarrow \quad L_+ L_-^k w = \left(2k\lambda - k(k-1)\right) L_-^{k-1} w$$

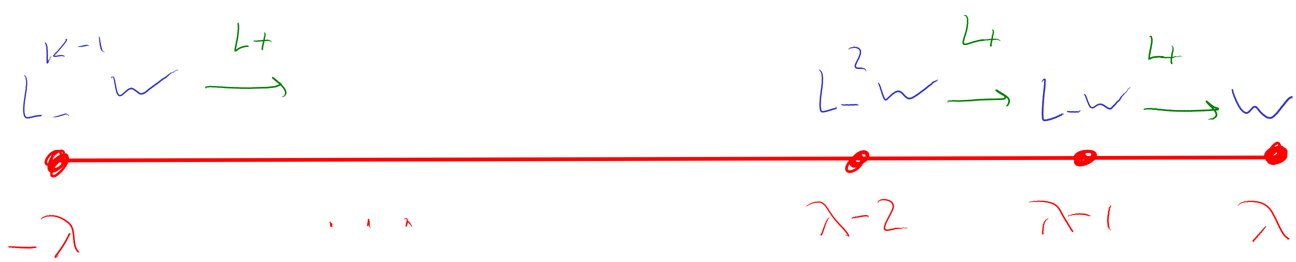
But <u>finite</u> $\Rightarrow L_-^k w = 0$ for some k

$$\Rightarrow \quad 2k\lambda - k(k-1) = 0$$

$$\Rightarrow \quad \boxed{\lambda = \frac{k-1}{2}} \in \frac{\mathbb{Z}}{2}\,! \qquad \text{"spin"}$$

& $\Rightarrow \quad k = 2\lambda + 1$

$\therefore \quad \lambda, \lambda - 1, \ldots, \lambda - (2\lambda) = -\lambda$



<u>Weight diagram</u> of any irrep of $su(2)$

su(3)

Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

basis for $su(3) \otimes \mathbb{C}$

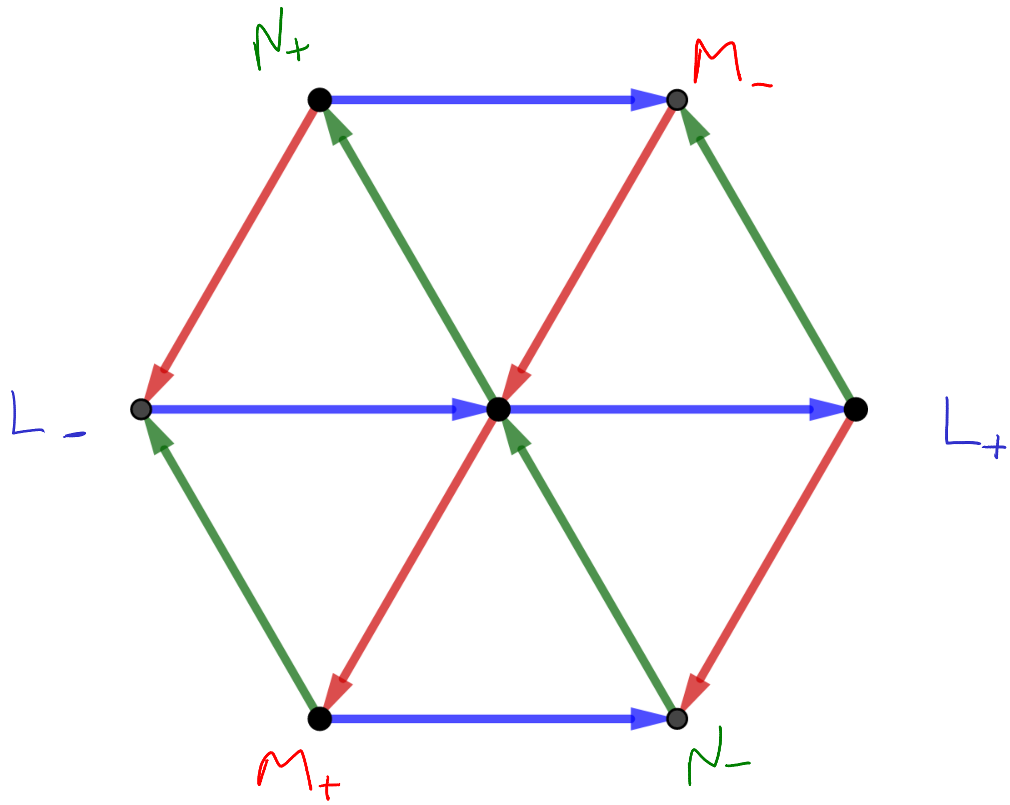
$(\mu_m = -i\lambda_m$ is basis for $su(3))$

$\longmapsto$ $sl(3,\mathbb{R}) \longleftrightarrow \lambda_1, \mu_2, \lambda_3, \lambda_4, \mu_5, \lambda_6, \mu_7, \lambda_8$

$\Rightarrow$ simultaneous eigenvectors of λ_3, λ_8 are

		λ_3	λ_8
$2L_\pm$	$\lambda_1 \pm i\lambda_2$	± 2	0
$2M_\pm$	$\lambda_4 \pm i\lambda_5$	∓ 1	$\mp\sqrt{3}$
$2N_\pm$	$\lambda_7 \pm i\lambda_8$	∓ 1	$\pm\sqrt{3}$
	λ_3	0	0
	λ_8	0	0

Root Diagram of su(3)



origin is $\underline{\underline{2\text{-}d}}$ vector space spanned by λ_3, λ_8

Weight Diagram of $su(3)$ rep

Consider defining rep of $su(3)$:

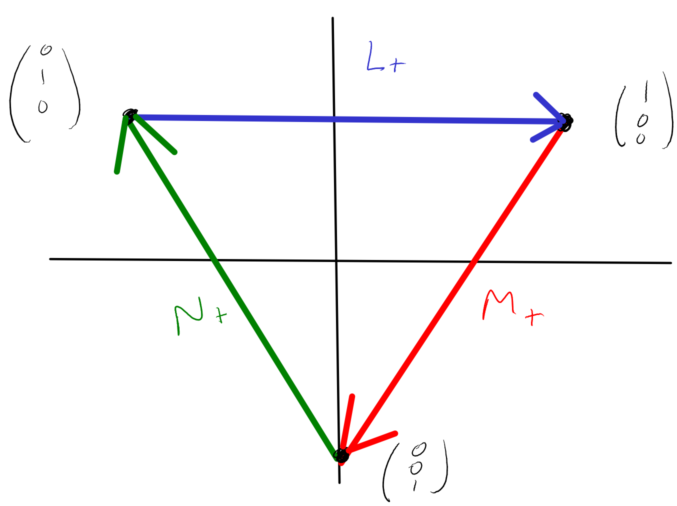
$$\boxed{\begin{gathered}\{ A \in \mathbb{C}^{3\times 3} : A^\dagger + A = 0,\ \mathrm{tr}\,A = 0 \} \\ \text{acting on } \mathbb{C}^3\end{gathered}}$$

As before, complexify & choose 2 commuting elements

λ_3, λ_8

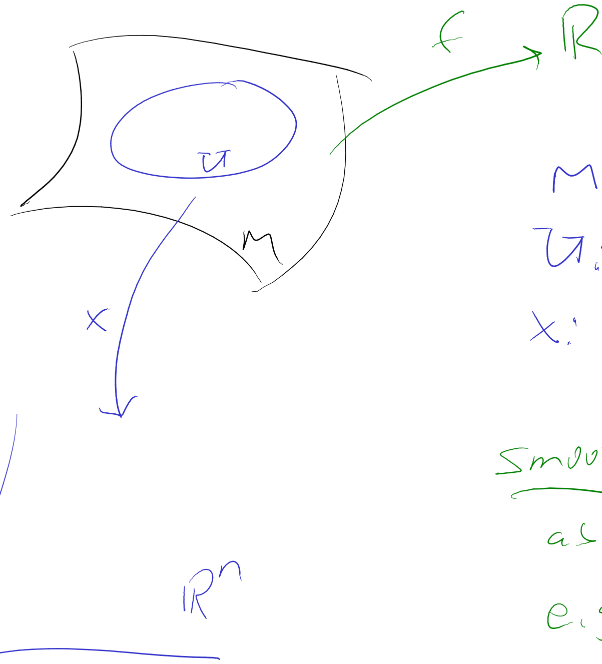
Now, find eigenvectors & eigenvalues

	λ_3	λ_8
$\begin{pmatrix}1\\0\\0\end{pmatrix}$	1	$1/\sqrt{3}$
$\begin{pmatrix}0\\1\\0\end{pmatrix}$	-1	$1/\sqrt{3}$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$	0	$-2/\sqrt{3}$



Differential Geometry

Idea:



M: topological space

U: open subset

x: coordinates ("chart")

smooth means smooth as maps on $\mathbb{R}^n$

e.g. $f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}$

vector fields $\longleftrightarrow$ directional derivatives (w.r.t. coordinates)

Left-Invariant Vector Fields

Day 07

Idea: Given Lie group G

$\mapsto$ Lie algebra $\mathfrak{g}$ is $T_{\mathbb{1}} G$

$\parallel$

tangent space at $\mathbb{1} \in G$

But $M \in G \Rightarrow M : G \to G$

$\mathbb{1} \mapsto M$

"push-forward"

"differential"

$\Rightarrow M_* : T_{\mathbb{1}} G \to T_M G$

$\therefore$ get vector field from any vector $A \in \mathfrak{g}$

$$\boxed{X|_M = M_* A}$$

$\Rightarrow \quad M_* X = X$ left-invariant vector field

"nice" curves

$\Rightarrow$ integral curves through $\mathbb{1}$ of (left-invariant) vector fields satisfy

$$\boxed{M(\alpha) = e^{A\alpha}}$$

1-parameter family

Properties of Killing form

Recall: $\cdot\ B(Y, X) = B(X, Y)$ symmetric

$\cdot\ B([X,Y],Z) = B([Y,Z],X)$ cyclic

$$\Rightarrow B([Z,X],Y) + B(X,[Z,Y]) = 0$$

invariant under G

$\Rightarrow$ B is constant on left-invariant vector fields!

Connection

Given a metric B on a Lie group G $\exists!$ Levi-Civita connection, that is, a derivative operator ∇_Z on vector fields:

- $\nabla_Z f = Z(f)$ (directional derivative)
- $\nabla_Z(B(X,Y)) = B(\nabla_Z X, Y) + B(X, \nabla_Z Y)$ (metric compatible)
  $= Z(B(X,Y)) = 0$
- $\nabla_X Y - \nabla_Y X = [X,Y]$ (torsion free)

But $B(X,Y) = \text{const}$ (for left-invariant vector fields)!

$$\Rightarrow B(\nabla_X Y, Z) = B([X,Y],Z) + B(\nabla_Y X, Z)$$

$$= B([X,Y],Z) - B(X, \nabla_Y Z)$$

$$= B([X,Y],Z) - B(X,[Y,Z]) - B(X, \nabla_Z Y)$$

$$= B([X,Y],Z) - B(X,[Y,Z]) + B(\nabla_Z X, Y)$$

$$= B([X,Y],Z) - B([Y,Z],X) + B([Z,X],Y) - B(\nabla_X Z, Y)$$

$$= B([X,Y],Z) - \cancel{B([Y,Z],X)} + \cancel{B([Z,X],Y)} + B(Z, \nabla_X Y)$$

(cyclic!)

$$\Rightarrow 2B(\nabla_X Y, Z) = B([X,Y],Z) \quad \forall Z$$

$$\Rightarrow \boxed{\nabla_X Y = \frac{1}{2}[X,Y]}$$

for left-invariant vector fields

Curvature

Given a connection ∇_Z, the curvature operator is

$$\boxed{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}$$

Here, $\nabla_X Y = \frac{1}{2}[X,Y]$, so

$$R(X,Y)Z = \frac{1}{4}\Big[X,[Y,Z]\Big] - \frac{1}{4}\Big[Y,[X,Z]\Big] - \frac{1}{2}\Big[[X,Y],Z\Big]$$

Jacobi identity

$$\Longrightarrow \;= \frac{1}{4}\Big[[X,Y],Z\Big] - \frac{1}{2}\Big[[X,Y],Z\Big]$$

so $$\boxed{R(X,Y)Z = -\frac{1}{4}\Big[[X,Y],Z\Big]}$$

for left-invariant vector fields

Ex: $so(3)$, basis $e_a = \{r_x, r_y, r_z\}$

$$\Rightarrow R(u,v)w = R^a{}_{bcd}\, w^b u^c v^d e_a$$

components of Riemann tensor

$$\Rightarrow R(r_x,r_y)r_z = 0$$
$$R(r_x,r_y)r_x = -\frac{1}{4} r_y$$

$$\Rightarrow R^a{}_{zxy} = 0, \quad R^y{}_{xxy} = -\frac{1}{4} \Rightarrow R^{xy}{}_{xy} = +\frac{1}{4}$$

$$\Rightarrow R^x{}_x = \frac{1}{2} = R^y{}_y = R^z{}_z \Rightarrow R = 3/2 = \text{const} \checkmark$$

Ricci tensor: $R^a{}_b = R^{ma}{}_{mb}$

Ricci scalar: $R = R^m{}_m = \mathrm{tr}\, Ric$