Spinors

Recall: $so(3) \leftrightarrow$ 3x3 real, antisymmetric

acts on: 3-component real column vectors

"vectors"

$su(2) \leftrightarrow$ 2x2 complex, antihermitian

acts on: 2-component complex column vectors

"spinors"

$$\mathbb{R}^3 \leftrightarrow \langle \sigma_m \rangle \longleftrightarrow \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = \underline{X}$$

$$M \in SU(2): \quad \underline{X} \mapsto M \underline{X} M^\dagger$$

$$\Rightarrow A \in su(2): \quad \underline{X} \mapsto [A, \underline{X}]$$

$\therefore$ <u>$SU(2)$ is double cover of $SO(3)$</u>

$\therefore$ <u>vectors</u> <u>and</u> Lie group/algebra are matrices

σ_m $\qquad$ $s_m = -i\,\sigma_m$

But $\sigma_x \sigma_y = i\,\sigma_z$.

What other products are there?

$$\mathbb{1} = \sigma_m^2 \quad ; \quad i\,\mathbb{1} = \sigma_x \sigma_y \sigma_z$$

Clifford Algebras

Given
- vector space V
- nondegenerate inner product g

$\Rightarrow$ Clifford algebra $Cl(V)$:

add bilinear, associative product

$$X^2 = |X|^2 = g(X,X)$$

$$\Longleftrightarrow \{X, Y\} = 2g(X,Y)$$

$$\{X,Y\} = XY + YX$$

anticommutator

symmetric

Pf: $(X+Y)^2 = \cancel{X^2} + \cancel{Y^2} + \{X, Y\}$

$$(X+Y)^2 = g(X+Y, X+Y) = \cancel{|X|^2} + \cancel{|Y|^2} + 2g(X,Y)$$

Orthonormal Basis:

$$\{\gamma_i, \gamma_j\} = \pm 2\,\delta_{ij}$$

gamma matrices!

signature!

$$V = \mathbb{R}^{p,q} \Longrightarrow Cl(V) = Cl(p,q)$$

① $\langle \gamma_i \rangle \cong V \subset Cl(V)$ rank 1

② $\gamma_i^2 = \pm \mathbb{1} \in Cl(V)$ rank 0

③ $\gamma_{i \dots j} = \gamma_i \cdots \gamma_j$ basis for rank k
if $1 \leq i < \dots < j \leq n = p+q$

$\therefore$ $\binom{n}{k}$ elements of rank k

& dim $Cl(p+q)$ is n^2

Exterior Algebra

$\Lambda(\bar{V})$: bilinear, associative product

wedge

$$w \wedge v = -v \wedge w$$

antisymmetric

$v \wedge v = 0$

$\Rightarrow$ basis is $e_{i \ldots j} = e_i \wedge \cdots \wedge e_j$

for each rank $k \le n$

same size

Extend to 1 : $1 \wedge v = v$

rank 0

Again, $\bar{V} \subset \Lambda(\bar{V})$ as rank-1 elements

Ex: Differential forms $\leftarrow$ vector fields

$$df(\bar{X}) = \bar{X}(f) = \bar{X}^m e_m(f)$$

directional derivative

Clifford product

$\therefore$ define $a \vee b = a \wedge b + g(a,b)$

vee wedge dot

Clifford Rotations

Recall : $\mathbb{R}^{p,q} \subset Cl(p,q)$

$$V = v^m \gamma_m$$

Assume first that γ_i, γ_j are spacelike

$$\therefore \; R_{ij}(\alpha) : \quad \begin{aligned} \gamma_i &\longmapsto \gamma_i \cos\alpha - \gamma_j \sin\alpha \\ \gamma_j &\longmapsto \gamma_i \sin\alpha + \gamma_j \cos\alpha \\ \gamma_k &\longmapsto \gamma_k \qquad\qquad k \neq i, j \end{aligned}$$

$\cap$ $SO(p,q)$

$$\Rightarrow \; r_{ij} : \quad \boxed{\begin{aligned} \gamma_i &\longmapsto -\gamma_j \\ \gamma_j &\longmapsto \gamma_i \\ \gamma_k &\longmapsto 0 \end{aligned}}$$

If boost, use sinh & cosh (& drop minus sign)

Calculate :

$$[\gamma_i \gamma_j, \gamma_i] = -2|\gamma_i|^2 \gamma_j$$

$$[\gamma_i \gamma_j, \gamma_j] = +2|\gamma_j|^2 \gamma_i$$

$$[\gamma_i \gamma_j, \gamma_k] = 0$$

$$\therefore \; \boxed{r_{ij} \longleftrightarrow \text{commute with } \tfrac{1}{2}\gamma_i \gamma_j}$$

$\therefore$

rank 0 :	$\mathbb{1}$	scalars	$\mathbb{R}$
rank 1 :	γ_i	vectors	$\mathbb{R}^{p,q}$
rank 2 :	r_{ij}	rotations	$so(p,q)$

Geometric Algebra

So $so(p,q) \subset Cl(p,q)$.
what about $SO(p,q)$?
Exponentiate!!

$$e^{\frac{1}{2}\gamma_i\gamma_j\alpha} = \mathbb{1} + \frac{1}{2}\gamma_i\gamma_j\alpha + \frac{1}{4}(\gamma_i\gamma_j)^2\alpha^2 + \cdots$$

$$= \mathbb{1}\left(1 - |\gamma_i|^2|\gamma_j|^2\left(\frac{\alpha}{2}\right)^2 + \cdots\right)$$

$$+ \gamma_i\gamma_j\left(\frac{\alpha}{2} - |\gamma_i|^2|\gamma_j|^2\left(\frac{\alpha}{2}\right)^2 + \cdots\right)$$

$$= \begin{cases} \cos\frac{\alpha}{2} + \gamma_i\gamma_j \sin\frac{\alpha}{2} & (|\gamma_i|^2|\gamma_j|^2 = +1) \\ \cosh\frac{\alpha}{2} + \gamma_i\gamma_j \sinh\frac{\alpha}{2} & (|\gamma_i|^2|\gamma_j|^2 = -1) \end{cases}$$

$$= R_{ij}\left(\frac{\alpha}{2}\right)$$

$$\therefore \boxed{SO(p,q) \subset Cl(p,q)}\,!$$

Classification of Clifford Algebras

Idea: Find the smallest matrices that represent $Cl(p,q)$

- $Cl(p,q)$ is a real vector space
  (can also classify over $\mathbb{C}$)
- matrices can be real, complex, quaternionic

Vectors: copy of $\mathbb{R}^{p,q} \subset Cl(p,q)$

$$\{\gamma_m\}$$

Spinors: column vectors acted on by $Cl(p,q)$

Special cases:

- smallest real matrices have same size as smallest complex matrices
  $\longmapsto$ Majorana spinors

- "spinors" not irreducible rep of $so(p,q)$
  ($\Rightarrow \exists!$ 2 irreducible reps)
  $\longmapsto$ Weyl spinors

Example: $\{\gamma_m\}$ = orthonormal basis

$$\Longrightarrow \quad \begin{aligned} \gamma_m^2 &= +\mathbb{1} \qquad (m = 1 \dots p) \\ \gamma_m^2 &= -\mathbb{1} \qquad (m = p+1 \dots p+q = n) \end{aligned}$$

$$\omega = \gamma_1 \dots \gamma_m$$

$$\begin{aligned} \Longrightarrow \quad \omega^2 &= (\gamma_1 \dots \gamma_m)(\gamma_1 \dots \gamma_m) \\ &= (-1)^{n(n-1)/2} \gamma_1^2 \dots \gamma_m^2 \\ &= (-1)^{n(n-1)/2} (-1)^q \, \mathbb{1} \end{aligned}$$

But: $$\frac{n(n-1)}{2} + q = \frac{(p+q)(p+q-1) + 2q}{2}$$

$$= \frac{(p+q)^2 + q - p}{2} = \frac{4pq + (p-q)^2 + q - p}{2}$$

$$= \cancel{2pq} + \frac{(p-q)(p-q-1)}{2}$$

even

depends only on $p-q$

$$\Longrightarrow \quad \begin{aligned} \omega^2 &= \mathbb{1} \qquad (p - q \equiv 0, 1 \mod 4) \\ \omega^2 &= -\mathbb{1} \qquad (p - q \equiv 2, 3 \mod 4) \end{aligned}$$

correctly suggests that classification depends only on

$p - q$ <u>mod 8</u>

<u>Spinorial Chessboard</u>

Examples

$Cl(1,0)$: $\quad V = \langle \sigma_z \rangle$

$$\Rightarrow \quad Cl(1,0) = \mathbb{R} \oplus \mathbb{R}$$

$Cl(0,1)$: $\quad V = \langle s_y \rangle$

$$\Rightarrow \quad Cl(0,1) = \mathbb{C}$$

$Cl(1,0)$ revisited : $\quad V = \langle \sigma_x \rangle$

$$\Rightarrow \quad Cl(1,0) = \mathbb{C}'$$

$Cl(2,0)$: $\quad V = \langle \sigma_x, \sigma_z \rangle \Rightarrow \omega = s_y$

$$\Rightarrow \omega^2 = -\mathbb{1}$$

$$\Rightarrow \quad Cl(2,0) = \mathbb{R}^{2\times 2}$$

$Cl(1,1)$: $\quad V = \langle \sigma_z, s_y \rangle \Rightarrow \omega = \sigma_z$

$$\Rightarrow \omega^2 = +\mathbb{1}$$

swap

$\sigma_x \leftrightarrow s_y$!

$$\Rightarrow \quad Cl(2,0) = \mathbb{R}^{2\times 2}$$

$Cl(0,2)$: $\quad V = \langle s_x, s_y \rangle \Rightarrow \omega = s_z$

$$i \qquad j \qquad k$$

$$\therefore \quad Cl(0,2) = \mathbb{H} \quad !$$

Clifford Algebras over Division Algebras

Weyl representation of $C\ell(3,1)$

$$\gamma_t = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right) = -s_y \otimes \mathbb{1} = i\sigma_y \otimes \mathbb{1}$$

$$\gamma_m = \left(\begin{array}{c|c} & \sigma_m \\ \hline \sigma_m & \end{array}\right) = \sigma_x \otimes \sigma_m$$

$\Longrightarrow$ vectors (rank 1) are block off-diagonal

$\Longrightarrow$ adjoint (rank 2) is block diagonal

$\Longrightarrow$ 2 Weyl spinor representations "$\left(\frac{1}{2}\right)$"

Step 1: $x\sigma_x + y\sigma_y = \begin{pmatrix} 0 & x-iy \\ x+iy & 0 \end{pmatrix}$ $\in \mathbb{C}$

∴ generalize to $\mathbb{H}, \mathbb{O}$!

$$\longmapsto \quad X = \begin{pmatrix} 0 & \bar{a} \\ a & 0 \end{pmatrix} \qquad a \in \mathbb{O}$$

$$\gamma_X = \left(\begin{array}{c|c} 0 & X \\ \hline X & 0 \end{array}\right) \Longrightarrow \{\gamma_X, \gamma_Y\} \longleftrightarrow \{X, Y\} \longleftrightarrow 2a\cdot b$$

$SO(8)$!

Step 2: Can add γ_z, γ_t back in

$$\longmapsto \quad \underline{X} = \begin{pmatrix} t+z & \bar{a} \\ a & t-z \end{pmatrix} \qquad \underline{a \in \mathbb{O}}$$

$$\gamma_{\underline{X}} = \left(\begin{array}{c|c} 0 & \underline{X} \\ \hline \tilde{\underline{X}} & 0 \end{array} \right)$$

$$\tilde{\underline{X}} = \underline{X} - \operatorname{tr}(\underline{X})\,\mathbb{1}$$

Aside: 2×2 characteristic equation:

$$\underline{X}^2 - \underline{X}\operatorname{tr}\underline{X} + \det\underline{X}\,\mathbb{1} = 0$$

$$\Rightarrow \underline{X}\,\tilde{\underline{X}} = -\det\underline{X}\,\mathbb{1}$$

$$-\det\underline{X} = -t^2 + z^2 + |a|^2$$

$$\therefore \quad \begin{aligned} \mathbb{R} &\longleftrightarrow SO(2,1) \\ \mathbb{C} &\longleftrightarrow SO(3,1) \\ \mathbb{H} &\longleftrightarrow SO(5,1) \\ \mathbb{O} &\longleftrightarrow SO(9,1) \end{aligned}$$

Classical supersymmetry only works in 3, 4, 6, 10 spacetime dimensions!!

Step 3 : $t\gamma_t + z\gamma_z \longleftrightarrow \left(\begin{array}{c|c} & \begin{matrix} t+z & \\ & t-z \end{matrix} \\ \hline & \end{array}\right) \longleftrightarrow \begin{pmatrix} t+z & \\ & t-z \end{pmatrix}$

$\mapsto tL\gamma_t + z\gamma_z \longleftrightarrow \begin{pmatrix} z+tL & \\ & -z+tL \end{pmatrix} \in \mathbb{C}'$!

$\therefore$ generalize to $\mathbb{H}'$, $\mathbb{O}'$!

$$\mathbf{X} = \begin{pmatrix} A & 0 \\ 0 & -\bar{A} \end{pmatrix} \qquad A \in \mathbb{C}' \Rightarrow \tilde{\mathbf{X}} = \begin{pmatrix} \bar{A} & 0 \\ 0 & -A \end{pmatrix}$$

$$\gamma_{\mathbf{X}} = \left(\begin{array}{c|c} & \mathbf{X} \\ \hline \tilde{\mathbf{X}} & \end{array}\right) \Rightarrow \{\gamma_{\mathbf{X}}, \gamma_{\mathbf{Y}}\} \longleftrightarrow 2A \cdot B$$

$SO(4,4)$

Step 4 : Do both !!

$$\mathbf{X} = \begin{pmatrix} A & \bar{a} \\ a & -\bar{A} \end{pmatrix}, \quad a \in \mathbb{O},\ A \in \mathbb{O}'$$

$$\gamma_{\mathbf{X}} = \left(\begin{array}{c|c} & \mathbf{X} \\ \hline \tilde{\mathbf{X}} & \end{array}\right) \quad \left(-\det \mathbf{X} = |a|^2 + |A|^2\right)$$

$\mapsto \underline{\underline{SO(12,4)}}$

The 2x2 Magic Square

Recall: $\mathbb{X} = \begin{pmatrix} A & \bar{a} \\ a & -\bar{A} \end{pmatrix} = x^m \sigma_m$

$\Longrightarrow \mathbb{X} \in \mathbb{R}^{p,q}$

$a \in \mathbb{K} \leftrightarrow (k, 0)$

$A \in \mathbb{K}' \leftrightarrow \left(\frac{k'}{2}, \frac{k'}{2}\right)$

$p = k + \frac{k'}{2},\ q = \frac{k'}{2}$

$\longmapsto \Gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \tilde{\sigma}_m & 0 \end{pmatrix}$ generates $C\ell(p,q)$

$\therefore\ \Gamma_m \Gamma_n$ generates $so(p,q)$ two copies!

$\therefore\ \boxed{\sigma_m \sigma_n \text{ generates } so(p,q)}$

anti-Hermitian $\longmapsto$ $\underline{so(p,q) \cong su(2, \mathbb{K}'\otimes\mathbb{K})}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{K}$
$\mathbb{R}'$	$so(2)$	$so(3)$	$so(5)$	$so(9)$	$su(2,\mathbb{K})$
$\mathbb{C}'$	$so(2,1)$	$so(3,1)$	$so(5,1)$	$so(9,1)$	$sl(2,\mathbb{K})$
$\mathbb{H}'$	$so(3,2)$	$so(4,2)$	$so(6,2)$	$so(10,2)$	$sp(2,\mathbb{K})$
$\mathbb{O}'$	$so(5,4)$	$so(6,4)$	$so(8,4)$	$so(12,4)$	?

This is half-split case. Also have double-split & compact

Flips

- what does $q \mapsto e^{k\alpha/2}\, q\, e^{-k\alpha/2}$ do to $q \in \mathbb{H}$?
- what happens if $q \in \mathbb{O}$?

Answers: over $\mathbb{H}$, rotates $i\partial$ by α
over $\mathbb{O}$, rotates $i\partial$, $\partial\ell$, $i\ell$, ℓk!

Need flips!

Idea:
$$e^{k\alpha/2} = \cos\frac{\alpha}{2} + k\sin\frac{\alpha}{2}$$
$$= \left(i\cos\frac{\alpha}{2} + \partial\sin\frac{\alpha}{2}\right)(-i)$$

flips by $\frac{\pi}{2}$

Count: $su(2, \mathbb{K}'\otimes\mathbb{K})$

- start with: $X_{aA} = \begin{pmatrix} 0 & aA \\ -\overline{aA} & 0 \end{pmatrix}$

generates rotation in aA plane

- How many are there?
- How many products?

Spinors

Recall: $so(p,q)$ acts on vectors X as

$$X \mapsto [\Gamma_m \Gamma_n, X]$$
$$= \Gamma_m(\Gamma_n X) - X(\Gamma_m \Gamma_n)$$

spinors are just 2-component column vectors
over $\mathbb{K}' \otimes \mathbb{K}$!

$\therefore$ $\chi = \left(\begin{array}{c|c} X & \theta \\ \hline -\theta^+ & \Phi \end{array} \right)$ combines vector X with spinor θ and scalar Φ

Furthermore $\mathcal{M} = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array} \right)$ acts as

$$\mathcal{M} \chi \mathcal{M}^{-1} = \left(\begin{array}{c|c} M X M^{-1} & M\theta \\ \hline -\theta^+ M^+ & \Phi \end{array} \right)$$

vector

spinor

scalar

works well for representations in first two rows – but not otherwise

X has only $\mathbb{K}$ on off diagonal;
θ has $\mathbb{K}' \otimes \mathbb{K}$.
$\therefore$ doesn't close

3×3 Magic Square

Idea: $\boxed{su(3, \mathbb{K}' \otimes \mathbb{K})}$

∴ start with $X_{aA} = \begin{pmatrix} 0 & aA & 0 \\ -\overline{aA} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

& Y_{aA}, Z_{aA} cyclic

- How many?
- How many products?

Answer: over $\mathbb{H}$, need 3 extra diagonal elements as before

But over $\mathbb{O}$ only need 14, not 21

$$so(\mathrm{Im}\,\mathbb{H}) = so(3)$$
$$so(\mathrm{Im}\,\mathbb{O}) = so(7)$$

but

$$\mathrm{Aut}\,\mathbb{H} = SO(3)$$
$$\mathrm{Aut}\,\mathbb{O} = g_2 < SO(7)$$

Automorphisms

Want! $\boxed{\Phi(xy) = \Phi(x)\Phi(y)}$

$\Rightarrow \Phi(1) = 1$

$\mathbb{R}$: Done! Nothing to do!

$\mathbb{C}$: Only automorphism is $i \longmapsto -i$

$\mathbb{H}$: $x \longmapsto q x q^{-1}$

$\Rightarrow xy \longmapsto q x \cancel{q^{-1}} \cancel{q} y q^{-1}$

$= q x y q^{-1}$ ✓

$\therefore \operatorname{Aut} \mathbb{H} \longleftrightarrow$ unit quaternions

$\longleftrightarrow SO(3)$

$$\boxed{\operatorname{Aut} \mathbb{H} = SO(3)}$$

$\mathbb{O}$: $(q x q^{-1})(q y q^{-1}) \neq q(xy)q^{-1}$!

G_2

Count $\mathrm{Aut}(\mathbb{O})$:

① $i \longmapsto j$ 6 choices $S^6 \subset \mathbb{R}^7$

② $j \longmapsto l$ 5 choices $S^5 \subset \mathbb{R}^6$

③ $k \longmapsto jl$ 0 choices !

④ $l \longmapsto i$ 3 choices $S^3 \subset \mathbb{R}^4$

$$\therefore \dim \mathrm{Aut}(\mathbb{O}) = 14$$

Idea: Rotate $ij \Rightarrow$ must also rotate il jl and/or k kl

$$k = ij = (jl)(il) = l(kl)$$

Fact: $i \longmapsto j \longmapsto -i$

& $jl \longmapsto -il \longmapsto -jl$ opposite orientations

works

Count: 7 planes $\times$ 2 indpt rotations

$= 14$ ✓

Albert Algebra

Recall: $X = \begin{pmatrix} tL+z & \bar{a} \\ a & tL-z \end{pmatrix}$ $\in H_2(\mathbb{O})$ Hermitian

$\Longrightarrow -\det X = -t^2 + z^2 + |a|^2$

$so(k+1,1) \cong sl(2,\mathbb{K}) \cong su(2, \mathbb{C}' \otimes \mathbb{K})$

Try 3x3:

$$\mathcal{X} = \begin{pmatrix} p & \bar{a} & c \\ a & m & \bar{b} \\ \bar{c} & b & n \end{pmatrix} \in H_3(\mathbb{O})$$

Jordan algebra!

$$(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{X}^2 = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{X}^2)$$

needed for observables in quantum mechanics

via density matrices "$\sum |\psi\rangle\langle\psi|$"

But: need commutative product.

$\therefore$ Jordan product $\boxed{\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X})}$

Fact: all Jordan algebras except Albert algebra arise from Jordan product on associative algebra.

"exceptional Jordan algebra"

Properties

① $x^2 = x \circ x$

② $x^3 = x^2 \circ x = x \circ x^2$ by definition

(not defined otherwise)

③ Freudenthal product:

$$\boxed{x * y = x \circ y - \frac{1}{2}\left(x \operatorname{tr} y + y \operatorname{tr} x\right) + \frac{1}{2}\left(\operatorname{tr} x \operatorname{tr} y - \operatorname{tr}(x \circ y)\right)}$$

$\Rightarrow$ x satisfies characteristic equation

$\Longleftrightarrow$ $\boxed{\det x = \frac{1}{3} \operatorname{tr}\left((x * x) \circ x\right)}$

Idea: Consider $v, w \in \mathbb{R}^3$

$\Rightarrow$ $x = vv^t$, $y = ww^t$ $\in H_3(\mathbb{R})$

$\Rightarrow$

$$\boxed{\begin{aligned} \operatorname{tr}(x \circ y) &= (v \cdot w)^2 \\ x * y &= (v \times w)(v \times w)^t \end{aligned}}$$

"dot & cross products!"

F_4 and E_6

$F_4 = SU(3, \mathbb{O})$ preserves trace of $X \in H_3(\mathbb{O})$

$\cap$

$E_6 = SL(3, \mathbb{O})$ preserves determinant

Count : f_4 : $\mathbb{O}^3$ on off diagonal

$\longleftrightarrow 24$

$so(8)$ on 2×2 diagonal

$\longleftrightarrow 28$

Nothing more!!

$\therefore \dim f_4 = 52$

e_6 : $(\mathbb{C}' \otimes \mathbb{O})^3$ on off diagonal

$\longleftrightarrow 48$

$so(8)$ on diagonal over $\mathbb{O}$

$\longleftrightarrow 28$

λ_3, λ_8 on $\mathbb{C}'$ diagonal

$\longleftrightarrow 2$

$\therefore \dim e_6 = 78$

Decompositions

Given any subalgebra $h \subset g$, h acts on $h_\perp$

(nondegenerate)

Ex: $so(3) = so(2) \oplus \mathbb{R}^2$

S_z S_x, S_y

Lie alg   Lie subalg   representation of Lie subalg

Orthogonal

$$so(p+1) = so(p) \oplus \mathbb{R}^p$$

$$so(p+q) = so(p) \oplus so(q) \oplus \mathbb{R}^{pq}$$

Unitary

$$su(p+1) = su(p) \oplus \mathbb{C}^p \oplus u(1)$$

$$su(p+q) = su(p) \oplus su(q) \oplus \mathbb{C}^{pq} \oplus u(1)$$

E_8

Fact: smallest representation of e_8 is e_8...

Problem: How to act on nested elements?!

Instead: Decompose!

$$M \in SU(2, \mathbb{K}' \otimes \mathbb{K}), \quad \Theta \in (\mathbb{K}' \otimes \mathbb{K})^2$$

$$\boxed{\mathcal{M} = \left(\begin{array}{c|c} M & \Theta \\ \hline -\Theta^{+} & \alpha \end{array}\right)} \in SU(3, \mathbb{K} \otimes \mathbb{K}')$$

$\therefore$ $M \in so\left(\mathbb{K} + \frac{\mathbb{K}'}{2}, \frac{\mathbb{K}'}{2}\right)$ adjoint

Θ spinor of $so\left(\mathbb{K} + \frac{\mathbb{K}'}{2}, \frac{\mathbb{K}'}{2}\right)$

α ?? — but not needed over $\mathbb{O}$

$$\therefore \quad e_{8(-24)} = so(12,4) \oplus \mathbb{R}^{128}$$

contains

Majorara-Weyl spinor of $so(12,4)$

Standard model $\rightarrow$ $su(3) \oplus su(2) \oplus u(1)$

$\oplus$ $so(3,1)$

Lorentz